More induction

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We’ve seen in Chapter 5 of Eccles that induction is used in definitions and in proofs. Here I want to explain how it can be used also to derive formulas that we would prove using induction. Consider the sum of the squares of the first $n$ positive integers:

$$\sum_{i=1}^{n} i^2.$$

Imagining a pyramid of squares of cubes: $1 \times 1$ at the top, atop $2 \times 2$, atop $3 \times 3$, ..., atop $n \times n$, since the volume of a pyramid is $\frac{1}{3}bh$, where $b$ is the area of the base and $h$ is its height, we might guess that this sum should be something like $\frac{1}{3}n^2 \cdot n = \frac{1}{3}n^3$. We would guess the same thing by recognizing it as being analogous to the integral

$$\int_0^n x^2 \, dx = \frac{1}{3}n^3.$$

Our experience with the simpler sum

$$\sum_{i=1}^{n} i = \frac{n}{2}(n + 1),$$

however, suggests that a better guess for the sum should also have smaller powers of $n$ in it, e.g., $an^3 + bn^2 + cn + d$.

Suppose we were given such a formula, with specific values for $a$, $b$, $c$, and $d$. Let us proceed as we would if we were trying to prove its correctness. We would do so using induction, so the first thing to check is the base case:

$$P(0) : \sum_{i=1}^{0} i^2 = 0 = d,$$

which would be true if $d = 0$. This implies that for our formula to be correct, it must have $d = 0$. The next thing we would do is to assume

$$P(k) : \sum_{i=1}^{k} i^2 = ak^3 + bk^2 + ck$$
(no constant $d$ since we already found that it must be 0). Then, to prove

\[ P(k + 1) : \sum_{i=1}^{k+1} i^2 = a(k + 1)^3 + b(k + 1)^2 + c(k + 1), \]

we use the inductive definition of the sum on the left hand side to get

\[
\sum_{i=1}^{k+1} i^2 = (k + 1)^2 + \sum_{i=1}^{k} i^2
\]

\[
= (k + 1)^2 + ak^3 + bk^2 + ck,
\]

where the second equality follows from $P(k)$. In order for this last expression to be equal to the right hand side of $P(k + 1)$ above, the coefficients of each power of $k$ must be the same. That is,

\[
k^3 : \quad a = a
\]
\[
k^2 : \quad 3a + b = 1 + b \quad \Rightarrow a = 1/3
\]
\[
k^1 : \quad 3a + 2b + c = 2 + c \quad \Rightarrow b = 1/2
\]
\[
k^0 : \quad a + b + c = 1 \quad \Rightarrow c = 1/6.
\]

Thus the formula must be

\[
\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n}{6}(2n + 1)(n + 1).
\]

And we have already shown that it can be proved by induction!