Background for de Grey’s result on the chromatic number of the plane

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Math 109 Mathematical Reasoning
University of California, San Diego
La Jolla, CA, 11 April 2018
Nelson’s problem (1950)

Can the points of the plane be colored with three colors so that every equilateral triangle with sides of length one has one vertex of each color?
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So far, so good ... but we need to color all the points in the plane.
Seurat’s attempt (1884–1886)

Un dimanche après-midi à l’Île de la Grande Jatte
Seurat’s attempt, three-colorized

*Un dimanche après-midi à l’Ile de la Grande Jatte*
Seurat’s attempt, three-colorized

*Un dimanche après-midi à l’Ile de la Grande Jatte*

OK
Seurat’s attempt, three-colorized

Un dimanche après-midi à l’Île de la Grande Jatte

OK; OK
Seurat’s attempt, three-colorized

*Un dimanche après-midi à l’Ile de la Grande Jatte*

OK; OK; not OK
Van Gogh’s attempt (1889)

Starry night
Van Gogh’s attempt, three-colorized

*Starry night*
Van Gogh’s attempt, three-colorized

Starry night

OK
Van Gogh’s attempt, three-colorized

Starry night

OK; not OK
Pollock’s attempt (1950)

Lavender mist
Pollock’s attempt, three-colorized

Lavender mist
Pollock’s attempt, three-colorized

Lavender mist

OK
Pollock’s attempt, three-colorized

Lavender mist

OK; not OK
Proof of impossibility

Suppose the points of the plane can be three-colored so that every equilateral triangle with sides of length 1 has one vertex of each color:

Then every point a distance 1 from a red point must be green or blue.
Proof of impossibility (cont.)

Suppose the points of the plane can be three-colored so that every equilateral triangle with sides of length 1 has one vertex of each color:

And every point a distance $\sqrt{3}$ from a red point must be red . . . which implies that there are two red points a distance 1 apart, a contradiction.
Combinatorial graphs

A graph $G$ consists of a set $V$ of vertices and set $E$ of edges, each of which is a pair $\{u, v\}$, $u, v \in V$. 

![Graph Diagram]
Graph colorings

A $k$-coloring of a graph $G$ is a map $f : V \rightarrow C$, where $C$ is a set with $k \in \mathbb{N}$ elements (the colors), such that if $\{u, v\} \in E$ then $f(u) \neq f(v)$.

A 7-coloring.
Graph colorings

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Chromatic number

The chromatic number, $\chi(G)$, of a graph $G$ is the smallest $k \in \mathbb{N}$ such that $G$ has a $k$-coloring.

This graph has chromatic number 4, because it has no 3-coloring.
The Hadwiger-Nelson problem (1961)

Let $G$ be the infinite graph with all the points of the plane as vertices and edges $\{u, v\}$ for all pairs of points distance 1 apart. What is $\chi(G)$?
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Nelson (1950): $\chi(G) \geq 4$.

This is a corollary of our observation that there is no 3-coloring of the plane such that the vertices of every equilateral triangle with sides of length one have three different colors.
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Isbell (1950): $\chi(G) \leq 7$.

Proof by construction: (diameter 1 hexagons)
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de Grey (2018): $\chi(G) \geq 5$. 