Please write out your proofs in complete sentences, justifying how you get from step to step.

1. [10 points] Recall that the Fibonacci numbers are defined by $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \in \mathbb{Z}_{>1}$. Prove that

$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}.$$  

By induction: It is true for $n = 1$ since $f_1 = 1 = f_2 = f_{2\cdot1}$. Supposing it is true for $n = k$, then $f_1 + f_3 + \cdots + f_{2k-1} + f_{2(k+1)-1} = f_{2k} + f_{2k+1} = f_{2k+2} = f_{2(k+1)}$.  

Fibonacci numbers occur in nature—I sometimes discuss them in my mathematical modeling class, MATH 111—but they are also of interest in elementary number theory, which is the beginning of MATH 104. See what you can figure out about this question: Which Fibonacci numbers are divisible by a given prime number $p$?

2. Let $G = \{e, a, b, c\}$ be a set. We want to define a multiplication operation on $G$ that satisfies the following properties:

- $\forall g \in G$, $e$ times $g$ (which we write as $eg$) is $g$. Also, $ge = g$.
- $\forall g \in G$, $\exists h \in G$ such that $gh = e$ and also $hg = e$.
- $\forall f, g, h \in G$, $(fg)h = f(gh)$.

a. [10 points] Prove that each row, and each column, of the multiplication table to the right must contain each element of $G$ exactly once.

Suppose the $g$ row contained some element $f \in G$ more than once. Then there would be $g_1 \neq g_2 \in G$ (labeling the corresponding columns) such that $gg_1 = f = gg_2$. But there is some $h \in G$ such that $hg = e$, so left multiplying by $h$ gives $hgg_1 = hf = hgg_2$, which implies $g_1 = hf = g_2$, which is a contradiction. The argument for columns is the same, using right multiplication.

b. [5 points] Fill in the blank spaces in the multiplication table. (Don’t forget to copy it into your blue book!)

The set $G$ with the multiplication operation obeying these rules is called a group. You will learn about groups in MATH 100 or MATH 103. The multiplication table here is called a Cayley table. I encourage you to look at problem 2 on the other lecture’s final exam; the group there is the other finite group of size 4 (there are only 2). You may have noticed that you did not need to use the third property, associativity, to fill in the multiplication table. In case you’re interested, you can try filling in the table on the next page, for a 6 element group; for this one you will need to use associativity.
3. [15 points] Using the \((\epsilon, \delta)\)-definition of limit, prove that \(\lim_{x \to 4} \sqrt{x} = 2\).

Suppose \(\epsilon > 0\). Choose \(\delta = \min\{4, 2\epsilon\}\). If \(0 < |x - 4| < \delta \leq 4\) then \(x > 0\), so \(\sqrt{x}\) is real and \(|\sqrt{x} + 2| > 2\). Thus

\[
|\sqrt{x} - 2| = \left| \frac{\sqrt{x} - 2}{\sqrt{x} + 2} \right| \cdot \left| \sqrt{x} + 2 \right| = \frac{|x - 4|}{|\sqrt{x} + 2|} < \frac{|x - 4|}{2} < \frac{\delta}{2} \leq \epsilon.
\]

4. Suppose \(f : \mathbb{R} \to \mathbb{R}\) is defined by \(f(x) = e^x\).

a. [5 points] Is \(f\) an injection? Is \(f\) a surjection? Is \(f\) a bijection? Please prove your answers.

\(f\) is an injection since \(e^{x_1} = e^{x_2}\) implies \(x_1 = x_2\) by taking the logarithm of both sides. \(f\) is not a surjection since there is no \(x \in \mathbb{R}\) such that \(e^x = -1\), for example. This means \(f\) is not a bijection.

b. [5 points] Prove that \(f(x + y) = f(x)f(y)\).

\(f(x + y) = e^{x+y} = e^x e^y = f(x)f(y)\).

The transformation between addition and multiplication from exponentiating, and in the reverse, by taking logarithms, is more general than just when applied to the real numbers, and is an important idea in several parts of math you may learn in the next couple of years. You’ll study the exponential and logarithmic functions in complex analysis, MATH 120. You may have learned in linear algebra that the exponential of a square matrix, \(A\), is defined by the same infinite series as in the scalar case: \(e^A = I + A + A^2/2! + \cdots\). This is important for solving systems of linear ordinary differential equations, the subject of MATH 130. You might try proving that \(e^{A+B} = e^A e^B\), if \(AB = BA\). Matrix exponentiation also arises in differential geometry, MATH 150 and in Lie groups, MATH 251.
5. [15 points] Prove that for all \( n \in \mathbb{Z}_{\geq 0} \), \( \sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n \).

Using the binomial theorem, \( \sum_{k=0}^{n} 2^k \binom{n}{k} = (1 + 2)^n = 3^n \).

This is a (simple) example of a binomial coefficient identity. You may see more complicated ones in combinatorics, MATH 154, and in probability, MATH 180.

6. [15 points] Let \( \Sigma = \{a, b, c\} \). Let \( \Sigma^n = \underbrace{\Sigma \times \cdots \times \Sigma}_{n \text{ times}} \). Let \( \Sigma^* = \bigcup_{n \in \mathbb{Z}_{>0}} \Sigma^n \). Is \( \Sigma^* \) countable or uncountable? Please prove your answer.

\( \Sigma^* \) is countable because we can define a bijection \( \mathbb{Z}_{>0} \to \Sigma^* \) as follows: \( |\Sigma^n| = 3^n \), which is finite. So we list the elements of \( \Sigma^* \): first the 3 elements of \( \Sigma^1 \) in alphabetical order, second the 3\(^2\) elements of \( \Sigma^2 \) in alphabetical order, etc.

The ideas in problems 3 and 6 are the beginnings of analysis, MATH 140 or MATH 142, and topology, MATH 190 and MATH 191. Another amazing construction is the Cantor set (which also appears in MATH 130): From the unit interval \([0, 1]\), remove the middle third \((1/3, 2/3)\). Repeat this in each of the remaining thirds. Repeat infinitely. Show that the set of points that are never removed has the cardinality of \( \mathbb{R} \), even though the total length of the removed intervals is 1.

7. [10 points] Let \( a_0 = 0, a_1 = 2, \) and \( a_n = a_{n-1} + a_{n-2} \), for \( n \in \mathbb{Z}_{>1} \). What is \( \gcd(a_n, a_{n+1}) \)? Please prove your answer.

We proved in lecture (and in the book) that when \( a, b \in \mathbb{Z} \), \( b > 0 \) and \( 0 \leq r < b \) is the remainder when \( a \) is divided by \( b \), then \( \gcd(b, a) = \gcd(r, b) \). So we will prove our answer by induction on \( n \). First notice that for \( n = 0 \), \( \gcd(0, 2) = 2 \). Suppose that \( \gcd(a_k, a_{k+1}) = 2 \). Then, since \( a_{k+2} = 1 \times a_{k+1} + a_k \), with \( 0 \leq a_k < a_{k+1} \), we have \( \gcd(a_{k+1}, a_{k+2}) = \gcd(a_k, a_{k+1}) = 2 \).

8. [10 points] Find the least significant digit (i.e., the ones digit) in \( 3^{2018} \).

The least significant digit of \( a \in \mathbb{Z} \) is \( [a]_{10} \). \( [3^{2018}]_{10} = [3]_{10}^{2018} \), so we need to understand powers of \( [3]_{10} \): \( [3]^0_{10} = [1]_{10}, [3]^1_{10} = [3]_{10}, [3]^2_{10} = [9]_{10}, [3]^3_{10} = [7]_{10}, [3]^4_{10} = [1]_{10} \), and the sequence repeats. Thus \( [3]^{2018}_{10} = [3]^{2}_{10} = [9]_{10} \) since \( 2018 \equiv 2 \mod 4 \), and the least significant digit of \( 3^{2018} \) is 9.

The last two problems use ideas from elementary number theory. MATH 104 goes from here to Diophantine equations (some of the chapters in Eccles we didn’t cover discuss the simplest cases), to prime numbers, and beyond.