Each problem is worth 25 points. Please be specific in your answer to problem 1, and please write out your proofs for the propositions in problems 2, 3 and 4 in complete sentences, justifying how you get from step to step.

1. What is the converse of this statement: “If pigs had wings, they could fly.”? Is the converse true? Why or why not?

   We can rewrite this statement as “pigs have wings ⇒ pigs can fly”, so its converse is “pigs can fly ⇒ pigs have wings”, which we would phrase in English as “If pigs could fly, they would have wings.”. Since pigs cannot fly (on their own!), the hypothesis of this implication is false, which means the statement is true. If you think pigs can fly in helicopters and thus the statement is false because they don’t have wings, I will accept that answer also. [See J. Withals, *A Dictionarie in English and Latine* (London: Thomas Purfoot 1616) p. 583 for what may be the origin of this proverb: “Pigs fly in the ayre with their tayles forward.”.]

2. Define a relation, ⪯, between pairs of real numbers: \((a, b) ⪯ (c, d)\) if \(a < c\) and \(b < d\). Prove that ⪯ is transitive.

   Let \(a, b, c, d, e, f \in \mathbb{R}\) with \((a, b) ⪯ (c, d)\) and \((c, d) ⪯ (e, f)\). By the definition of ⪯, this means that \(a < c, b < d, c < e,\) and \(d < f\). By the transitivity of <, this implies that \(a < e\) and \(b < f\). But this means \((a, b) ⪯ (e, f)\); hence ⪯ is transitive. [< is an example of a partial ordering, for which the trichotomy principle is not true.]

3. René-Jules wants to design a necklace of 17 beads made from coral (red) or ivory (white), arranged so that adjacent beads are different colors. Prove that he will not succeed.

   Suppose he succeeds, i.e., there is a necklace with 17 beads arranged so that adjacent beads are different colors. Pick a coral bead and pair it with one of the ivory beads next to it. This defines a direction along the necklace. Continue in this direction, pairing each coral bead with the next bead, which must be ivory. After 8 pairs there will be 1 remaining bead, which must be coral. But it is adjacent to the bead from which we started, which is also coral. This is a contradiction, so René-Jules must fail. [This is René-Jules Lalique, known for his naturalistic jewelry designs.]

4. Prove that \(\sum_{i=1}^{n}(2i - 1) = n^2\) for integers \(n \geq 0\).

   We will prove this by induction. The base case is \(P(0): \sum_{i=1}^{0}(2i - 1) = 0 = 0^2\), which is true. Assume \(P(k): \sum_{i=1}^{k}(2i - 1) = k^2\). To prove \(P(k + 1)\), compute:
   
   \[
   \sum_{i=1}^{k+1}(2i - 1) = 2(k + 1) - 1 + \sum_{i=1}^{k}(2i - 1) \quad \text{by the def of } \sum
   \]
   
   \[
   = 2k + 1 + k^2 \quad \text{by } P(k)
   \]
   
   \[
   = (k + 1)^2. \quad \text{by the distributive property}
   \]

   But this is \(P(k + 1)\), so the proposition is true. [So the sum of the first \(n\) odd numbers is \(n^2\), a square number. Can you draw this?]