Each problem is worth 25 points. Please be specific in your answer to problem 1, and please write out your proofs for the propositions in problems 2, 3 and 4 in complete sentences, justifying how you get from step to step.

1. Let $S$ be the statement: “If Trump feels insecure he tweets something stupid.” and let $T$ be the contrapositive of $S$. What is $T$? Is $S \text{XOR} T$ true or false? Why?

Since $S$ is the implication “Trump feels insecure $\Rightarrow$ he tweets something stupid”, its contrapositive $T$ is “He does not tweet something stupid $\Rightarrow$ Trump does not feel insecure”, which we would phrase in English as, “If Trump doesn’t tweet something stupid he is feeling secure.”. Since a statement and its contrapositive are equivalent, $S$ and $T$ have the same truth value. Although we might not all agree what this truth value is, we all (to the extent we all are logical!) must therefore agree that $S \text{XOR} T$ is false.

2. Prove the Boolean distribution law: $P \land (Q \lor R) \iff (P \land Q) \lor (P \land R)$.

We could prove this with a truth table, but let’s make a different argument. This is an iff statement, so we can prove both implications to prove the statement.

$(\Rightarrow)$ $P \land (Q \lor R)$ true means that $P$ must be true and either $Q$ or $R$ is true. If $P$ and $Q$ are true, then $(P \land Q) \lor (P \land R)$ since the first clause is true, and if $P$ and $R$ are true, then it is true since the second clause is true. Thus $P \land (Q \lor R) \Rightarrow (P \land Q) \lor (P \land R)$.

$(\Leftarrow)$ $(P \land Q) \lor (P \land R)$ means both $P$ and $Q$ are true or both $P$ and $R$ are true. Thus $P$ is true, and either $Q$ or $R$ is true, i.e., $P \land (Q \lor R)$. Thus $P \land (Q \lor R) \iff (P \land Q) \lor (P \land R)$.

3. Prove that no three unit vectors $u, v, w \in \mathbb{R}^2$ are mutually perpendicular.

We will prove this by contradiction, i.e., by assuming that there are three such vectors $u, v, w \in \mathbb{R}$. By rotating $\mathbb{R}^2$ we can choose $u = (1, 0)$. Then $u \perp v$ implies $v = (0, b)$, with $b \in \{-1, 1\}$. Let $w = (x, y)$. Then $u \perp w$ implies $x = 0$ and $v \perp w$ implies $y = 0$. But then the length of $w$ is 0, not 1, which is a contradiction. Thus there is no such set of vectors.

4. Prove that the number of ways to tile an $n \times 1$ rectangle with $1 \times 1$ squares and $2 \times 1$ rectangles is $f_{n+1}$, the $(n + 1)^{st}$ Fibonacci number.

We prove this using (strong) induction. First, consider $n = 1$. It can be tiled in 1 way, using a single $1 \times 1$ tile; this is $f_2$. Second, consider $n = 2$. It can be tiled in 2 ways, using two $1 \times 1$ tiles, or one $2 \times 1$ tile; this is $f_3$. Now suppose a $k \times 1$ rectangle can be tiled in $f_{k+1}$ ways, and also that a $(k - 1) \times 1$ rectangle can be tiled in $f_k$ ways (since we are using strong induction). Consider a $(k + 1) \times 1$ rectangle, tiled. The last tile must be either a $1 \times 1$ tile or a $2 \times 1$ tile. In the first case there will be $f_{k+1}$ tilings since that is the number of ways the remaining $k \times 1$ rectangle can be tiled, according to the inductive hypothesis, and in the second case there will be $f_k$ tilings since that is the number of ways the remaining $(k - 1) \times 1$ rectangle can be tiled, also according to the (strong) inductive hypothesis. Thus the total number of tilings is $f_{k+1} + f_k$, which is $f_{k+2}$ according to the definition of the Fibonacci numbers.