HW #1

Sec 1.1

#3: (a) $u_t - u_{xx} + 1 = 0$; 2nd order, linear inhomogeneous: PDE has form $\mathcal{L}u = -1$ where $\mathcal{L}$ is the linear operator $\mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$.

(b) $u_t - u_{xx} + xu = 0$; 2nd order, linear homogeneous: have form $\mathcal{L}u = 0$ where $\mathcal{L}$ is the linear operator $\mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x \cdot Id$, (where Id is the identity operator $Id(u) = u$).

(c) $u_t - u_{xxt} + uu_x = 0$; 3rd order, nonlinear: The operator is $\mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^3}{\partial t \partial x^2} + (Id)(\frac{\partial}{\partial x})$ (by this we mean that $\mathcal{L}u = u_t - u_{xxt} + (u)(u_x)$). If $u = x$ and $c = 2$ then

$$\mathcal{L}(cu) = \mathcal{L}(2x) = 0 - 0 + (2x)(2) = 4x, \text{ whereas}$$

$$c\mathcal{L}(u) = 2\mathcal{L}(x) = 2[0 - 0 + (x)(1)] = 2x, \text{ so}$$

$$\mathcal{L}(cu) \neq c\mathcal{L}(u) \Rightarrow \mathcal{L} \text{ is not linear.}$$

(d) $u_{tt} - u_{xx} + x^2 = 0$; 2nd order, linear inhomogeneous: have form $\mathcal{L}u = -x^2$ where $\mathcal{L} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ is linear.

(e) $iu_t - u_{xx} + \frac{1}{x}u = 0$; 2nd order, linear homogeneous: have form $\mathcal{L}u = 0$ where $\mathcal{L}$ is the linear operator $\mathcal{L} = i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{1}{x} Id$

(f) $u_x (1+(u_x)^2)^{-1/2} + u_y (1+(u_y)^2)^{-1/2} = 0$; 1st order, nonlinear: we may write the operator as

$$\mathcal{L} = \frac{\left(\frac{\partial}{\partial x}\right)}{\left[1+\left(\frac{\partial}{\partial x}\right)^2\right]^{1/2}} + \frac{\left(\frac{\partial}{\partial y}\right)}{\left[1+\left(\frac{\partial}{\partial y}\right)^2\right]^{1/2}}.$$ Let $u = x$, $c = 2$

Then $\mathcal{L}(2x) = \dfrac{(2)}{[1+(2)^2]^{1/2}}$, whereas

$2\mathcal{L}(x) = 2\left(\dfrac{(1)}{[1+(1)^2]^{1/2}}\right)$, so $\mathcal{L}$ is not linear.

(g) $u_x + e^y u_y = 0$; first order, linear homogeneous: have form $\mathcal{L}u = 0$ where $\mathcal{L}$ is the linear operator $\mathcal{L} = \frac{\partial}{\partial x} + e^y \frac{\partial}{\partial y}$.

(h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$; 4th order, non-linear: the operator is $\mathcal{L} = \frac{\partial}{\partial t} + \frac{\partial^4}{\partial x^4} + \sqrt{1+Id}$.

Set $u = 1$, $v = 0$. Then $\mathcal{L}(u+v) = \mathcal{L}(1)$

$= 0 + 0 + \sqrt{1+1} = \sqrt{2}$, whereas $\mathcal{L}(u) + \mathcal{L}(v) =$

$= \mathcal{L}(1) + \mathcal{L}(0) = \sqrt{2} + (0 + 0 + \sqrt{1+0}) = \sqrt{2} + 1$

So $\mathcal{L}(u+v) \neq \mathcal{L}(u) + \mathcal{L}(v) \Rightarrow \mathcal{L}$ is not linear.

#11: Let f + g be any differentiable functions of one variable and set $u(x,y) = f(x)g(y)$. Then

$$u_x = f'(x)g(y), \quad u_y = f(x)g'(y)$$

$$u_{xy} = f'(x)g'(y), \quad \text{and}$$

$$u\,u_{xy} = f(x)g(y)f'(x)g'(y) = u_x u_y \quad \text{as desired.}$$

#12: Note: $\sinh(x)$ is the hyperbolic sine function $\sinh(x) := \dfrac{e^x - e^{-x}}{2}$ (pronounced "sinch"). We also have the hyperbolic cosine function $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. Clearly $\dfrac{d}{dx}\sinh(x) = \cosh(x)$ and $\dfrac{d}{dx}\cosh(x) = \sinh(x)$. Set

$$u_n(x,y) = \sin(nx)\sinh(ny) \quad \text{for any } n > 0.$$

We show directly that u_n satisfies $u_{xx} + u_{yy} = 0$:

$$\frac{\partial}{\partial x}u_n = n\cos(nx)\sinh(ny),$$

$$\frac{\partial^2}{\partial x^2}u_n = -n^2\sin(nx)\sinh(ny),$$

$$\frac{\partial}{\partial y}u_n = n\sin(nx)\cosh(ny),$$

$$\frac{\partial^2}{\partial y^2}u_n = n^2\sin(nx)\sinh(ny),$$

so $$\frac{\partial^2}{\partial x^2}u_n + \frac{\partial^2}{\partial y^2}u_n = 0$$

$\boxed{\text{Sec } 1.2}$

#4: We check that $u(x,y) = f(e^{-x}y)$ is a solution to the PDE $u_x + y u_y = 0$.

By the chain rule,

$$u_x = f'(e^{-x}y)(-e^{-x}y), \quad u_y = f'(e^{-x}y)(e^{-x})$$

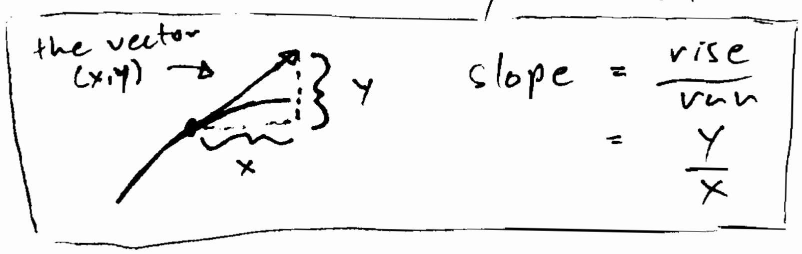
so

$$u_x + y u_y = (-e^{-x}y)f'(e^{-x}y) + (ye^{-x})f'(e^{-x}y) = 0$$

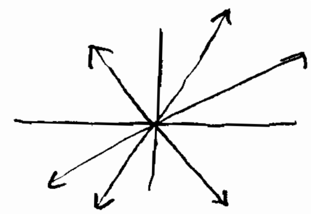
#5: We solve $xu_x + yu_y = 0$ using the geometric method. The PDE may be written as $\nabla u \cdot (x,y) = 0$. The quantity on the left is the directional derivative of u in the direction of the vector (x,y). Since the derivative is zero, u is flat (i.e. constant) as we move in this direction.

This suggests that we should find the curves that are always ~~tangent~~ have a tangent line described by the vector (x,y) at each point, AKA the characteristic curves. They must satisfy

$$\frac{dy}{dx} = \frac{y}{x}$$



Solving the ODE (for example by separation of variables) yields $y = cx$ where $c \in \mathbb{R}$ is arbitrary.



Hence the characteristic curves are the family of radial lines: They are parametrized by the value c. Since u is constant on each line, we have $u = f(c)$ (u depends only on which line one considers, and each line ~~depends~~ is determined by the value c.). Solving for c gives $c = \frac{y}{x}$, so $u = f(\frac{y}{x})$

<u>#8</u>: We solve $au_x + bu_y + cu = 0$. Make a change of coordinates

$s = ax + by$, $t = -bx + ay$. By the Chain Rule,

$$u_x = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = u_s(a) + u_t(-b)$$

$$u_y = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = u_s(b) + u_t(a)$$

Then $au_x + bu_y + cu = (a^2+b^2)u_s + (-ab+ab)u_t + cu$ so in the new coordinates the PDE becomes $u_s + \left(\frac{c}{a^2+b^2}\right)u = 0$. We can solve this using separation of variables or integrating factor: e.g. let $\mu = e^{\left(\frac{c}{a^2+b^2}\right)s}$. Then $\frac{\partial}{\partial s}(\mu \cdot u) = \mu \cdot u_s + \mu\left(\frac{c}{a^2+b^2}\right)u = 0$; integrating with respect to s gives $\mu u = f(t)$, so

$$u = \frac{f(t)}{\mu} = f(t)\, e^{\frac{-c}{a^2+b^2} s} = f(-bx+ay)\, e^{\frac{-c}{a^2+b^2}(ax+by)}$$

where f is arbitrary.

#9: We solve $u_x + u_y = 1$. Make the change of coordinates $s = x+y$, $t = x-y$. By the Chain Rule,

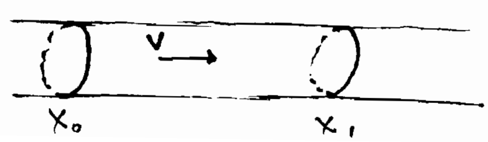
$$u_x = (u_s)(1) + (u_t)(1), \quad u_y = (u_s)(1) + (u_t)(-1),$$

(for more detail, see #8 above) so $u_x + u_y = (1+1)u_s + (1-1)u_t$, and the PDE becomes $2u_s = 1$ or $u_s = \frac{1}{2}$. Integrating, we find $u = \frac{1}{2}s + f(t)$; substituting for s and t gives

$$u = \tfrac{1}{2}(x+y) + f(x-y)$$

$\boxed{\text{Sec } 1.3}$

#5: The derivation is almost identical to that of the diffusion equation in the text (make sure you understand that first). We consider a section of pipe from x_0 to x_1:



If $u(x,t)$ is the concentration of the diffusing substance at position x and time t, then the mass in the interval $[x_0, x_1]$ at time t is $M(t) = \int_{x_0}^{x_1} u(x,t)\,dx$, and then $\frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x,t)\,dx$. We compute $\frac{dM}{dt}$ again by different means. The mass in this section of pipe can't change except by flowing in or out of its ends. The net rate of change of mass $\frac{dM}{dt}$ is given by $\frac{dM}{dt}$ = flow in − flow out.

There are now two phenomena that cause a flow of mass across the ends. The first is the diffusion of the substance; as we have seen, by Fick's Law this contributes a quantity $k u_x(x_1,t) - k u_x(x_0,t)$. The second is the movement of the fluid, which is moving at a speed v to the right. The rate at which the mass is flowing past the point x_0 (the "flow in") is (velocity) × (density at point x_0) $= v \cdot u(x_0,t)$. Similarly, the flow past x_1 (the "flow out") is $v \cdot u(x_1,t)$. So the movement of the fluid contributes a term

$v\,u(x_0,t) - v\,u(x_1,t)$. Hence

$$\frac{dM}{dt} = \left(\kappa\, u_x(x_1,t) - \kappa\, u_x(x_0,t)\right) + \left(v\,u(x_0,t) - v\,u(x_1,t)\right)$$

Equating the expressions for $\frac{dM}{dt}$ and differentiating w.r.t. x_1 gives

$$u_t(x_1,t) = \kappa\, u_{xx}(x_1,t) - v\, u_x(x_1,t)$$

i.e. $\quad u_t = \kappa\, u_{xx} - v\, u_x$

#6: We may assume that the specific heat and density of the cylinder are constant, so the heat equation reduces to the diffusion equation: $u_t = \kappa \Delta u$. The temperature depends only on the cylindrical coordinate $r = \sqrt{x^2+y^2}$, which suggests we should use the cylindrical-coordinate form of the Laplace operator: $\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} u + \frac{\partial^2 u}{\partial z^2}$

Note that u is constant when θ and z vary, so we have $\frac{\partial^2}{\partial \theta^2} u = \frac{\partial^2}{\partial z^2} u = 0$ and

$$u_t = \kappa \Delta u = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + 0 + 0\right]\kappa$$

$$= \left[\frac{1}{r}\left(r \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right)\right]\kappa = \left(u_{rr} + \frac{u_r}{r}\right)\kappa.$$

#8 Note: we will use the following notation to indicate triple integrals: $\int_D f \, d\underline{x}$ means $\iiint_D f \, dx\,dy\,dz$, where D is a region in $\mathbb{R}^3$.

Set $f(t) = \int_{\mathbb{R}^3} |u|^2 \, d\underline{x}$. We know that $f(0) = 1$; if we can show that f is a constant then it follows that $f(t) = 1$ for all t, as desired. To show that $f =$ constant, it suffices to show that $f'(t) = 0$. Note that u is complex-valued, and $|u|^2 = u\bar{u}$. Then

$$f'(t) = \frac{\partial}{\partial t}\int_{\mathbb{R}^3} |u|^2 \, d\underline{x} = \int_{\mathbb{R}^3} \frac{\partial}{\partial t} |u|^2 \, d\underline{x}$$

$$= \int_{\mathbb{R}^3} \frac{\partial}{\partial t} u\bar{u} \, d\underline{x} = \int_{\mathbb{R}^3} u_t \bar{u} + u \bar{u}_t \, d\underline{x} .$$

Now, u satisfies Schrödinger's Eqn:

$$(-i)h u_t = \frac{h^2}{2m}\Delta u + \frac{e^2}{r} u$$

Write $-i = \frac{1}{i}$ and solve for u_t:

$$u_t = \frac{ih}{2m}\Delta u + \frac{ie^2}{rh} u$$

taking the complex conjugate gives

$$\overline{u_t} = -\frac{ih}{2m}\Delta\bar{u} - \frac{ie^2}{rh}\bar{u}$$

Plugging in for u_t and $\overline{u_t}$ gives

$$f'(t) = \int_{\mathbb{R}^3} \left(\frac{ih}{2m}\Delta u + \frac{ie^2}{rh} u\right)\bar{u} + u\left(-\frac{ih}{2m}\bar{\Delta u} - \frac{ie^2}{rh}\bar{u}\right) d\underline{x}$$

$$\frac{ih}{2m} \int_{\mathbb{R}^3} \Delta u \bar{u} - \Delta \bar{u} u \, d\underline{x}$$

Note that $\nabla \cdot (\nabla u \bar{u} - \nabla \bar{u} u)$

$$= \nabla \cdot (\nabla u \bar{u}) - \nabla \cdot (\nabla \bar{u} u)$$

(Product rule)

$$= (\Delta u \bar{u} + \nabla u \nabla \bar{u}) - (\Delta \bar{u} u + \nabla \bar{u} \nabla u)$$

$$= \Delta u \bar{u} - \Delta \bar{u} u.$$

Letting $B(R)$ denote the ball of radius R centered at the origin $\left(= \{\underline{x} \in \mathbb{R}^3 : |\underline{x}| \leq R\}\right)$, we have

$$f'(t) = \frac{ih}{2m} \int_{\mathbb{R}^3} \nabla \cdot (\nabla u \bar{u} - \nabla \bar{u} u) \, d\underline{x}$$

$$= \lim_{R \to \infty} \frac{ih}{2m} \int_{B(R)} \nabla \cdot (\nabla u \bar{u} - \nabla \bar{u} u) \, d\underline{x}$$

By the Divergence Thm,

$$\iiint_{B(R)} \nabla \cdot (\nabla u \bar{u} - \nabla \bar{u} u) \, dx\,dy\,dz$$

$$= \iint_{\partial B(R)} (\nabla u \bar{u} - \nabla \bar{u} u) \cdot \vec{n} \, dS$$

According to the problem, we may assume that u and $\nabla u \to 0$ "fast enough" as $|\underline{x}| \to \infty$. Hence the quantity in the last integral is approaching zero as $R \to \infty$, so that $f'(t) = 0$.

To make this last step more rigorous and to see what "fast enough" means, we can do the following. Let

$G(R) := \max \{ |(\nabla u \bar{u} - \nabla \bar{u} u) \cdot \vec{n}|$ on the sphere $\partial B(R) \}$. Then

$$\iint_{\partial B(R)} (\nabla u \bar{u} - \nabla \bar{u} u) \cdot \vec{n} \, dS$$

$$\leq \text{Area}(\partial B(R)) \cdot G(R).$$

The area of the sphere increases at a rate proportional to R^2. Thus, it suffices to assume that $u, \nabla u \to 0$ fast enough so that the quantity $(\nabla u \bar{u} - \nabla \bar{u} u)$ decreases faster than $\frac{1}{R^2}$ as $R \to \infty$.