HW #2

Sec 1.4

#1: We solve $u_t = u_{xx}$, $u(x,0) = x^2$ by sheer force of will (i.e. trial + error): The initial condition suggests that u has form $u = x^2 + f(t)$ where $f(0) = 0$. Then $u_t = f'(t)$ and $u_{xx} = 2$ so $f'(t) = 2 \Rightarrow f(t) = 2t$ and we see that $u = x^2 + 2t$ does the job.

#3: After a long time we expect the heat to diffuse evenly throughout the body since no heat can enter or leave the system; hence the steady-state temperature is $u = C_0$, a constant.

Since heat is conserved we have

$$\begin{pmatrix}\text{Heat in body}\\ \text{at time } t=0\end{pmatrix} = \begin{pmatrix}\text{Heat in body}\\ \text{after a long time}\end{pmatrix}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\iiint_D c \cdot \rho \cdot f \, dx\,dy\,dz = \iiint_D c \cdot \rho \cdot C_0 \, dx\,dy\,dz$$

The body is assumed homogeneous so the specific heat c and density ρ are constants. Cancelling them gives

$$\iiint_D f\,dx\,dy\,dz = \iiint_D C_0\,dx\,dy\,dz$$
$$= C_0 \cdot \mathrm{Vol}(D)$$

$$\Rightarrow \quad C_0 = \frac{\left(\iiint_D f\,dx\,dy\,dz\right)}{\mathrm{Vol}(D)} = \text{Average value of } f \text{ on } D,$$

as we would expect.

$\boxed{\text{Sec 1.5}}$

#1: Consider $\frac{d^2}{dx^2}u + u = 0$ with boundary conditions $u(0) = u(L) = 0$. The ODE has general solution $u = A\sin x + B\cos x$, $A, B \in \mathbb{R}$. $u(0) = 0 \Rightarrow 0 = u(0) = 0 + B\cos(0) = B$, so u has form $u(x) = A\sin x$. Also $u(L) = 0$ $\Rightarrow 0 = u(L) = A\sin L$. If $L \neq \pi \cdot k$ for some integer k then $\sin L \neq 0$, and we can cancel this term to obtain $A = 0$, so $u(x) \equiv 0$. If $L = \pi \cdot k$ for some integer k then $A\sin L = 0$ regardless of A. In particular, $u(x) \equiv 0$ is the unique solution to the ODE + boundary value problem if and only if L is not an integer multiple of π.

#4: Consider the Neumann problem

$$\Delta u = f(x,y,z) \quad \text{in } D, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D$$

(where ∂D = boundary of D).

a.: Suppose u is a solution and c is a constant. Then

$\rightarrow \Delta(u+c) = \Delta u + \Delta c = f + 0$, and

$\rightarrow \frac{\partial}{\partial n}(u+c) = \frac{\partial u}{\partial n} + \frac{\partial}{\partial n} c = 0 + 0$ on ∂D,

so $u+c$ is also a solution. Hence the solution is not unique.

b.: We have

$$\iiint_D f \, dx\,dy\,dz = \iiint_D \nabla \cdot (\nabla u) \, dx\,dy\,dz$$

$$\text{(Div. Thm)} \quad = \iint_{\partial D} (\nabla u) \cdot \vec{n} \, dS$$

$$= \iint_{\partial D} \frac{\partial u}{\partial n} \, dS \overset{\left(\text{since } \frac{\partial u}{\partial n} = 0 \text{ on } \partial D\right)}{=} 0.$$

c. First, let's understand the implications of the Neumann condition $\frac{\partial u}{\partial n} = 0$ on ∂D. In terms of heat flow, this means that no heat can flow past the boundary of D, i.e. the region D is perfectly insulated. For diffusion, this means that the diffusing substance cannot flow past the boundary, i.e. the region D is bounded by some imperm- eable substance.

In part a) we saw that if u is a solution to the Neumann problem, then so is $u + c$. We can interpret this in terms of heat flow as follows. Suppose we observe the temperature using two thermometers, and suppose the scale on the second thermometer has been shifted so that it reads $10°$ above the actual temperature. Then we obtain readings u_1 and $u_2 = u_1 + 10$, which are both solutions to the Neumann problem. So adding a constant corresponds to changing one's reference point for temperature.

In part b) we saw that the existence of a solution requires that f satisfies $\iiint_D f\,dx\,dy\,dz$. One possible explanation of why this is so is as follows. Diffusion is governed by the equation $\frac{1}{k} u_t = \Delta u$. Hence we have $\frac{1}{k} u_t = \Delta u = f$

$$\Rightarrow \iiint_D f\,dx\,dy\,dz = \iiint_D \frac{1}{k} u_t\,dx\,dy\,dz$$

$$= \frac{1}{k}\frac{d}{dt} \iiint_D u\,dx\,dy\,dz.$$

This last quantity is the rate of change of total mass in D, which we expect to be zero since the mass is conserved.

#5: Consider $u_x + y u_y = 0$, w/ boundary condition $u(x,0) = \phi(x)$.

~~__~~ This PDE is solved in the text (pg #8); it has solution $u(x,y) = f(e^{-x} \cdot y)$. Note that ~~__~~ u must be constant on the line $y = 0$ since $u(x,0) = f(0)$. We conclude that

a. No solution exists for $\phi(x) = x$, since this implies that $u(x,0) = x$, contradicting that u is constant on the line $y = 0$.

b. There are many solutions for $\phi = 1$. In this case, we must have

$$1 = \phi(x) = u(x,0) = f(e^{-x} \cdot 0) = f(0),$$

so any function $f(w)$ such that $f(0) = 1$ will do the job. E.g. $f_1(w) = x^2 + 1$, $f_2(w) = \cos(w)$, $f_3(w) = e^w$ would all produce solutions.

$\boxed{\text{Sec } 2.1}$

#2: $u_{tt} = c^2 u_{xx}$, $u(x,0) = \log(1+x^2)$, $u_t(x,0) = 4+x$

~~__~~ In order to use d'Alembert's formula (pg 36), note that $F(w) = 4w + \frac{1}{2}w^2$ is an antiderivative of $\psi(w) = 4 + x$. According to the formula,

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

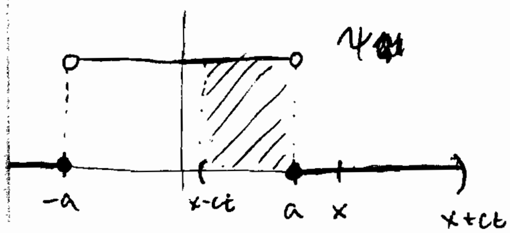
$$= \frac{1}{2}\left[\log(1+(x+ct)^2) + \log(1+(x-ct)^2)\right] + F(x+ct) - F(x-ct)$$

$$= \frac{1}{2}\log\left[(1+(x+ct)^2)(1+(x-ct)^2)\right] + 8ct + \frac{1}{2}\left[(x+ct)^2 - (x-ct)^2\right]$$

$$= \qquad '' \qquad + 8ct + 2ctx.$$

#5 We have $\phi = 0$ and $\psi(x) = \begin{cases} 1 & |x| < a \\ 0 & \text{else} \end{cases}$, so

$$u(x,t) = 0 + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds = \begin{pmatrix}\text{area under graph of } \psi \\ \text{between } x \mp ct \text{ and } x+ct\end{pmatrix}\frac{1}{2c}$$



The integral of ψ

$$= (\text{length}) \times (\text{height})\frac{1}{2c}$$

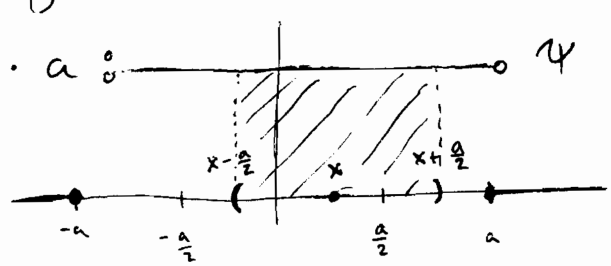
$$= \frac{1}{2c}\left(\text{length } (x-ct, x+ct) \cap (-a,a)\right) \times (1)$$

We draw the graph of u versus x for several fixed values of t.

$\boxed{t = \frac{a}{2c}}$ We have $u(x, \frac{a}{2c}) = \frac{1}{2c}\text{length } (x - \frac{a}{2}, x + \frac{a}{2}) \cap (-a,a)$

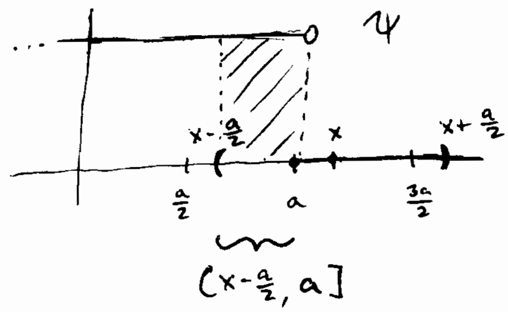
For $0 \le x \le \frac{a}{2}$, the interval $(x - \frac{a}{2}, x + \frac{a}{2})$ is contained in $(-a,a)$, so $\frac{1}{2c}\text{ length } (x - \frac{a}{2}, x + \frac{a}{2}) \cap (-a,a)$

$= \frac{1}{2c}\text{ length } (x - \frac{a}{2}, x + \frac{a}{2}) = \frac{1}{2c} \cdot a$:

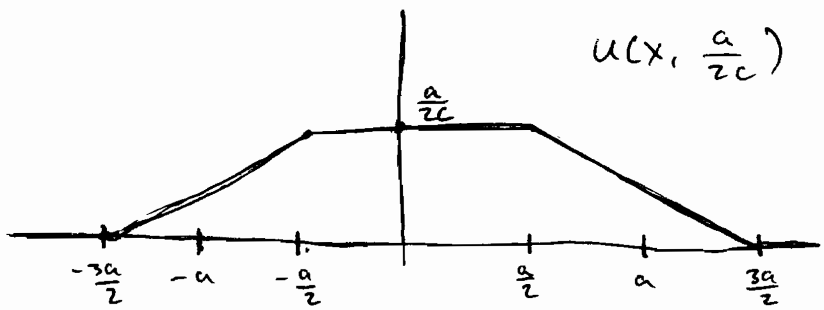


For $\frac{a}{2} \le x \le \frac{3a}{2}$, the intersection of $(x-\frac{a}{2}, x+\frac{a}{2})$ and $(-a, a)$ is an interval with left endpoint $x - \frac{a}{2}$ and right endpoint a, hence length $(x-\frac{a}{2}, x+\frac{a}{2}) \cap (-a, a]$

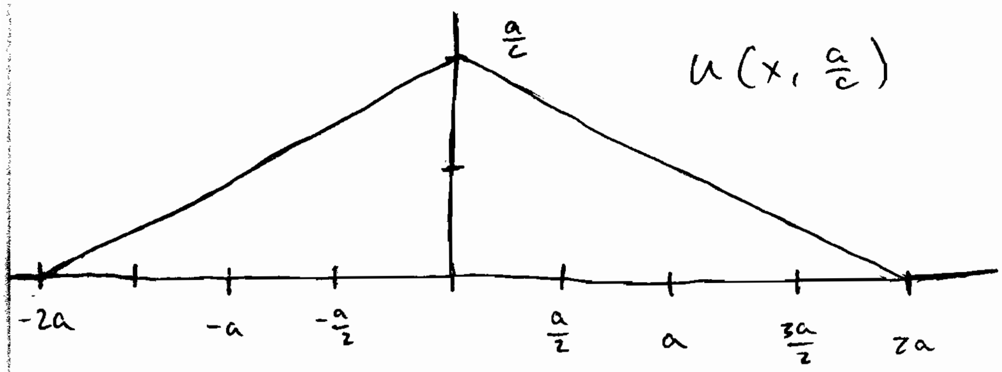
$$= a - (x - \tfrac{a}{2}) = \tfrac{3a}{2} - x \,:$$

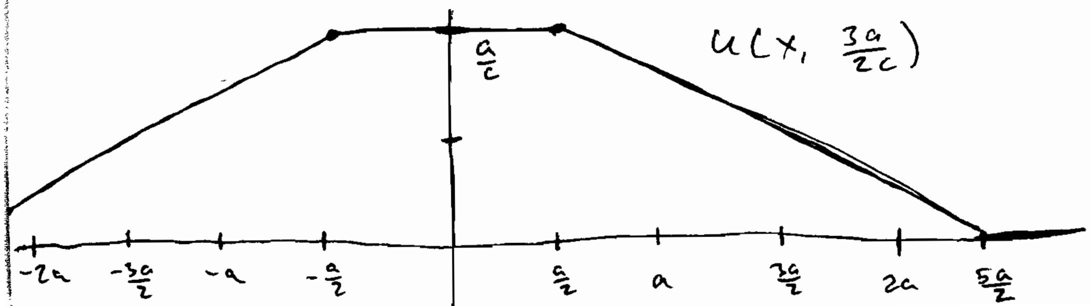


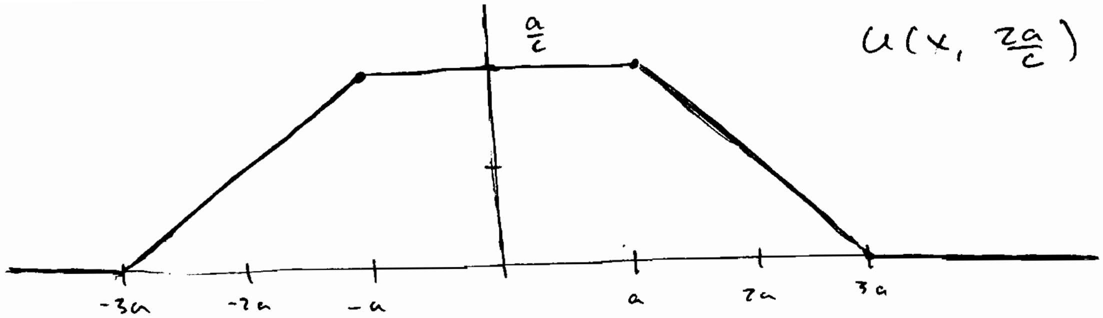
For $x > \frac{3}{2}a$, the intersection $(x-\frac{a}{2}, x+\frac{a}{2}) \cap (-a, a]$ is empty and hence it has length zero. So we have computed $u(x, \frac{a}{2c})$ for $x \ge 0$; since ψ is an even function, $u(x, \frac{a}{2c})$ is symmetric about the y-axis. The graph of $u(x, \frac{a}{2c})$ is

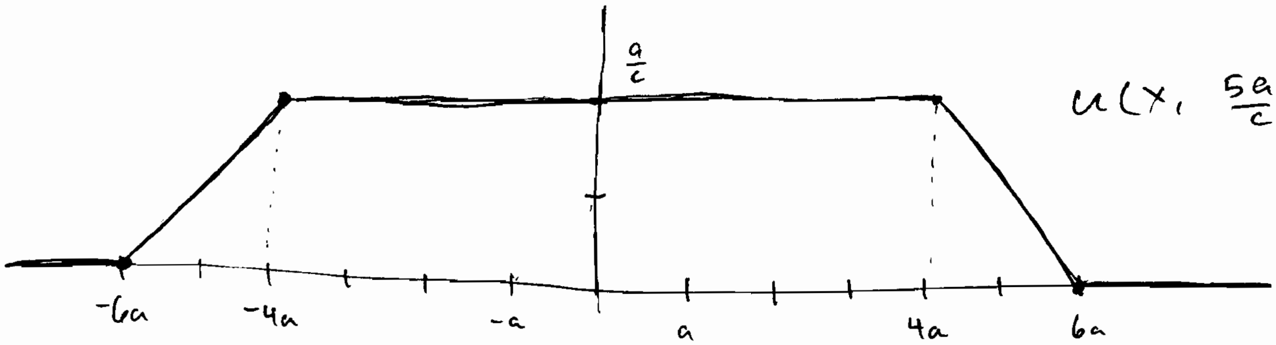


The other graphs are computed similarly.









#8 a: Let $v = ru$. We wish to show that $v_{tt} = c^2 v_{rr}$, given that $u_{tt} = c^2\left(u_{rr} + \frac{2}{r} u_r\right)$.

We have: $v_{tt} = \frac{\partial^2}{\partial t^2}(ru) = r u_{tt}$, and

$$v_r = \frac{\partial}{\partial r}(ru) = u + r u_r \quad \Rightarrow$$

$$v_{rr} = \frac{\partial}{\partial r}(u + r u_r) = u_r + u_r + r u_{rr}$$

$$= r u_{rr} + 2 u_r, \quad \text{so}$$

$$v_{tt} = r u_{tt} = r\left(c^2\left(u_{rr} + \frac{2}{r} u_r\right)\right) = c^2(r u_{rr} + 2u_r)$$

$$= c^2 v_{rr}.$$

b. $v_{tt} = c^2 v_{rr}$ is the wave equation; it has general solution $v(r,t) = f(r+ct) + g(r-ct)$.

c.: We solve for $u(r,t)$ given initial conditions $u(r,0) = \phi(r)$ and $u_t(r,0) = \psi(r)$. Define functions $\hat{\phi}$ and $\hat{\psi}$ by $\hat{\phi}(w) = w \cdot \phi(w)$, $\hat{\psi}(w) = w \cdot \psi(w)$. Then

$$v(r,0) = r \cdot u(r,0) = r \cdot \phi(r) = \hat{\phi}(r),$$
$$v_t(r,0) = r \cdot u_t(r,0) = r \cdot \psi(r) = \hat{\psi}(r).$$

We can then write down the formula for v in terms of $\hat{\phi}$ and $\hat{\psi}$:

$$v = \frac{1}{2}\left[\hat{\phi}(r+ct) + \hat{\phi}(r-ct)\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} \hat{\psi}(s)\,ds$$

Since $v = r \cdot u$, we can divide by r and substitute for $\hat{\phi}$ and $\hat{\psi}$ to get u in terms of ϕ and ψ:

$$u = \frac{1}{2r}\left[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)\right] + \frac{1}{2rc}\int_{r-ct}^{r+ct} s\,\psi(s)\,ds$$

#10: We solve $u_{xx} + u_{xt} - 20u_{tt} = 0$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$.

First, factor the differential operator:

$$0 = u_{xx} + u_{xt} - 20u_{tt} = \left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + 5\frac{\partial}{\partial t}\right)u$$

Next, make the change of coordinates $w = x + \frac{1}{4}t$, $z = x - \frac{1}{5}t$. By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial w} + \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial t} = \frac{1}{4}\frac{\partial}{\partial w} - \frac{1}{5}\frac{\partial}{\partial z}.$$

Substituting these expressions gives the new PDE $\left(\frac{9}{5}\frac{\partial}{\partial z}\right)\left(\frac{9}{4}\frac{\partial}{\partial w}\right)u = 0$ or, equivalently $\frac{\partial}{\partial z}\frac{\partial}{\partial w}u = 0$. Integrating first with respect to z and then with respect to w gives $u(w,z) = F(w) + G(z)$

$$\Rightarrow u(x,t) = F\left(x + \tfrac{1}{4}t\right) + G\left(x - \tfrac{1}{5}t\right) \qquad (*)$$

We have

$$\phi(x) = u(x,0) = F(x) + G(x)$$

$$\psi(x) = u_t(x,0) = \tfrac{1}{4}F'(x) - \tfrac{1}{5}G'(x)$$

Differentiating the first equation gives

$$\phi'(x) = F'(x) + G'(x).$$

Solving for F' and G' gives

$$F'(x) = \tfrac{4}{9}\left(\phi'(x) + 5\psi(x)\right)$$

$$G'(x) = \tfrac{5}{9}\left(\phi'(x) - 4\psi(x)\right)$$

Integrating yields

$$F(x) = \tfrac{4}{9}\phi(x) + \tfrac{20}{9}\int_0^x \psi(s)\,ds + A$$

$$G(x) = \tfrac{5}{9}\phi(x) - \tfrac{20}{9}\int_0^x \psi(s)\,ds + B$$

Since $\phi(x) = F(x) + G(x) = \phi(x) + 0 + (A+B)$, we have $A + B = 0$. Substituting these expressions for F & G in equation $(*)$ and combining the integrals gives

$$u(x,t) = \tfrac{4}{9}\phi\left(x + \tfrac{1}{4}t\right) + \tfrac{5}{9}\phi\left(x - \tfrac{1}{5}t\right) + \tfrac{20}{9}\int_{x-\frac{1}{5}t}^{x+\frac{1}{4}t}\psi(s)\,ds.$$