HW #3 Sol-ns

Sec 2.2

#3: Suppose u solves the wave equation $u_{tt} = c^2 u_{xx}$

Then so does:

<u>a.</u> any translate $u(x-y,t)$ for y fixed.

<u>Pf</u>: If $v(x,t) = u(x-y,t)$ then

$$v_{tt}(x,t) = u_{tt}(x-y,t) = c^2 u_{xx}(x-y,t) = c^2 v_{tt}(x,t)$$

<u>b.</u> any derivative of u.

<u>Pf</u>: E.g. if $v = u_x$, then (assuming u has has continuous partial derivatives up to order three) we have

$$(u_x)_{tt} = (u_{tt})_x = (c^2 u_{xx})_x = c^2 (u_x)_{xx}.$$

<u>c.</u> any dilation $u(ax, at)$ for a constant

<u>Pf</u>: If $v(x,t) = u(ax,at)$ then by the chain rule,

$$v_t(x,t) = u_t(ax,at) \cdot \frac{\partial(at)}{\partial t} = u_t(ax,at) \cdot a.$$

Likewise, $v_{tt}(x,t) = u_{tt}(ax,at)a^2$,

$$v_x(x,t) = u_x(ax,at) \cdot a$$

$$v_{xx}(x,t) = u_{xx}(ax,at) \cdot a^2, \quad \text{and}$$

$$v_{tt}(x,t) = a^2 u_{tt}(ax,at) = a^2(c^2 u_{xx}(ax,at))$$

$$= c^2(a^2 u_{xx}(ax,at)) = c^2 v_{xx}(x,t).$$

#5: For the damped string equation
$u_{tt} - c^2 u_{xx} + r u_t = 0$, $r > 0$,
we show that the energy decreases.
Note that $c = \sqrt{\left(\frac{T}{\rho}\right)}$ where T, ρ are constants. Further, we assume that u is is zero outside a bounded interval at time $t=0$. By causality, u is zero outside a (larger) bounded interval for any later time. Hence u_t, u_x are also zero for $|x|$ large enough.

We have $KE = \frac{1}{2}\rho\int_{-\infty}^{\infty} u_t^2\,dx$, so

$$\frac{dKE}{dt} = \frac{1}{2}\rho\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u_t^2\,dx$$

$$= \frac{1}{2}\rho\int_{-\infty}^{\infty} 2u_t u_{tt}\,dx \qquad \text{(using chain rule)}$$

$$= \rho\int_{-\infty}^{\infty} u_t\left(\left(\frac{T}{\rho}\right)u_{xx} - r u_t\right)u_t \qquad \text{(using PDE)}$$

$$= T\int_{-\infty}^{\infty} u_t u_{xx}\,dx - r\rho\int_{-\infty}^{\infty} u_t^2\,dx$$

(Int by parts)

$$= T\left(u_t u_x\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x u_{xt}\,dx\right) - r\rho\int_{-\infty}^{\infty} u_t^2\,dx$$

$$= 0 - \frac{T}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u_x^2\,dx - r\rho\int_{-\infty}^{\infty} u_t^2\,dx$$

(since u_t, u_x vanish for $|x|$ large, and $\frac{d}{dt}u_x^2 = 2u_x u_{xt}$)

$$PE = \frac{T}{2}\int_{-\infty}^{\infty} u_x^2\, dx, \quad \text{so}$$

$$\frac{dPE}{dt} = \frac{T}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u_x^2\, dx. \quad \text{Then}$$

$E = KE + PE \implies$

$$\frac{dE}{dt} = \frac{dKE}{dt} + \frac{dPE}{dt}$$

$$= \left(-\frac{T}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u_x^2\, dx - r\rho\int_{-\infty}^{\infty} u_t^2\, dx\right)$$

$$+ \left(\frac{T}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u_x^2\, dx\right)$$

$$= -r\rho\int_{-\infty}^{\infty} u_t^2\, dx \leq 0$$

since $r, \rho > 0$ and u_t^2 is nonnegative.

$\boxed{\text{Sec } 2.3}$

#2: Suppose u solves $u_t = u_{xx}$ in the strip $\{0 \leq x \leq l, 0 \leq t < \infty\}$.

a. Let $M(T)$ = max of $u(x,t)$ in the closed rectangle $\{0 \leq x \leq l, 0 \leq t \leq T\}$. Then M is an increasing function of T. To see this, suppose $T_1 \leq T_2$. The Maximum Principle says that for a rectangle of height T, u takes its max on the bottom or lateral sides. Let S_1 and S_2 denote the

union of these three sides for the rectangles of heights T_1 and T_2, respectively. Then

$M(T_1) = \max U(x,t)$ for points (x,t) in S_1,
$M(T_2) = \max U(x,t)$ for points (x,t) in S_2.

Since S_2 contains S_1, the second maximum must be at least as large as the first, so $M(T_1) \le M(T_2)$, i.e. M is increasing.

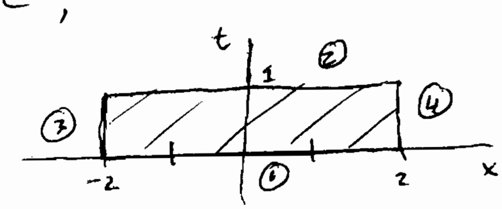
b. Let $m(T) = \cancel{\max u}$ min u ~~in the~~ in the rectangle of height T. Then m is decreasing: for $T_1 \le T_2$, define S_1 and S_2 as above. Then by the minimum principle,

$m(T_1) = \min U(x,t)$ for points (x,t) in S_1,
$m(T_2) = \min U(x,t)$ for points (x,t) in S_2

Again, S_2 contains S_1, so the second minimum is at least as small as the first, so $m(T_1) \ge m(T_2)$, i.e. m is decreasing.

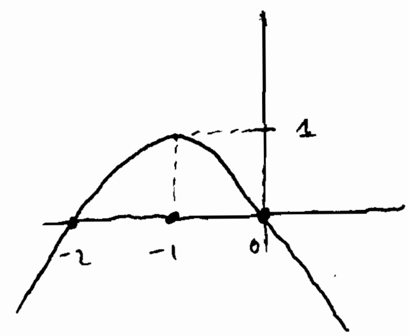
#5: We show that the max principle fails for the eqn $u_t = x u_{xx}$

a: Let $u = -2xt - \cancel{ax}\, x^2$. Note that $u_t = -2x$ and $u_{xx} = -2$, so $u_t = -2x = x u_{xx}$. We find the max of u on the rectangle $\{-2 \le x \le 2,\ 0 \le t \le 1\}$. To do this, we compute the max of u on the four sides of the rectangle, and on the interior.



① For the side $t=0$, we have $u(x,0) = -x^2$. For $-2 \le x \le 2$, the max value occurs at $x=0$; $u(0,0) = 0$.

② For the side $t=1$ we have $u(x,1) = -2x - x^2 = -x(2+x)$



This is a parabola opening downwards w/ zeros at $x=0$ and $x=-2$, so the max value occurs halfway in between, at $x=-1$. The max value is $u(-1,1) = 1$

③ For the side $x=-2$, $u(-2,t) = 4t-4$. For $0 \le t \le 1$, the max value occurs at $t=1$, and the max is $u(-2,1) = 0$.

④ For the side $x = 2$,
$u(2,t) = -4t - 4$, which has its maximum when $t = 0$; the max is $u(2,0) = -4$

For the interior, a (local) maximum can only occur at the critical points, i.e. points (x,t) satisfying $u_x(x,t) = 0$ and $u_t(x,t) = 0$.
We have $u_x = -2t - 2x$, $u_t = -2x$,
$u_t = 0 \iff x = 0$ and then $u_x = -2t - 2(0) = 0 \iff t = 0$. ~~So there is only~~
Since ~~the~~ $(0,0)$ is not an interior point, there are no critical points.

So the max of u on the rectangle ~~occurs~~ is the largest of the four values found in ① – ④, namely $u(-1,1) = \underline{1}$.
So the max does not occur on the three sides specified by the max principle.

b We show where the proof of the Maximum principle would break down if applied to the equation $u_t = x u_{xx}$. (You should

should glance at the proof for context). The proof examines the quantity $v_t - \kappa v_{xx}$. It is shown that $v_t - \kappa v_{xx} = -2\varepsilon\kappa$, and since $\varepsilon, \kappa > 0$, this yields the "diffusion ~~eq~~ inequality" $v_t - \kappa v_{xx} < 0$. If we replace κ with x, it is still true that $v_t - x v_{xx} = -2\varepsilon x$, but the latter quantity is only negative for $x > 0$, but we consider x in the range $-2 \le x \le 2$. So the diffusion inequality is no longer valid.

#6 See Midterm 1 solutions, problem #3.

$\boxed{\text{Sec } 2.4}$

#4 We solve $u_t = \kappa u_{xx}$ with initial condition $\phi(x) = e^{-x}$ for $x > 0$, $\phi(x) = 0$ for $x < 0$. We can apply the formula. Since $\phi = 0$ for $x < 0$ we have

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\kappa t}} \phi(y)\, dy$$

$$= \frac{1}{\sqrt{4\pi\kappa t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\kappa t}} e^{-y}\, dy.$$

Combining the exponential's and completing the square (see section 2.4 example #2) gives

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_0^\infty e^{-\frac{(y+2\kappa t-x)^2}{4\kappa t} + \kappa t - x}\, dy$$

Make the substitution $p = \frac{y + 2\kappa t - x}{\sqrt{4\kappa t}}$ to obtain $u(x,t) = \frac{1}{\sqrt{\pi}} e^{\kappa t - x} \int_{\frac{2\kappa t - x}{\sqrt{4\kappa t}}}^{\infty} e^{-p^2}\, dp$

This is apparently as far as we can go, although we can write the last integral in terms of the error function

$$\mathrm{Erf}(x) = \int_0^x e^{-p^2}\, dp :$$

$$\int_{\frac{2\kappa t - x}{\sqrt{4\kappa t}}}^{\infty} e^{-p^2}\, dp = \int_0^\infty e^{-p^2}\, dp - \int_0^{\frac{2\kappa t-x}{\sqrt{4\kappa t}}} e^{-p^2}\, dp$$

$$= \frac{\sqrt{\pi}}{2} - \mathrm{Erf}\left(\frac{2\kappa t - x}{\sqrt{4\kappa t}}\right)$$

Where the first integral is computed using exercise #6. Hence

$$u(x,t) = \frac{e^{\kappa t - x}}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} - \mathrm{Erf}\left(\frac{2\kappa t - x}{\sqrt{4\kappa t}}\right)\right)$$

<u>#9</u>: We solve $u_t = \kappa u_{xx}$, $u(x,0) = x^2$ $(*)$

Suppose u is a solution to $(*)$.

Then u_{xxx} also satisfies the diffusion equation (the proof is similar to exercise #3 of section 2.2). Moreover, since $u(x,0) = x^2$, differentiating three times shows that $u_{xxx}(x,0) = 0$. According to section 2.4, u_{xxx} is the unique solution to this new PDE + IC problem, so we must have $u_{xxx} = 0$ for any solution u of $(*)$. Integrating with respect to x gives $u_{xx} = A(t)$ for an arbitrary function $A(t)$. Integrating two more times gives

$$u = \frac{A(t)}{2} x^2 + B(t) x + C(t).$$

Then $u_t = \frac{A'(t)}{2} x^2 + B'(t) x + C'(t)$

$$u_{xx} = A(t)$$

and the diffusion equation gives

$$\frac{A'(t)}{2} x^2 + B'(t) x + C'(t) = k A(t).$$

This is an equality of polynomials in the variable x. Equating coefficients gives

$$\frac{A'(t)}{2} = B'(t) = 0, \quad C'(t) = k A(t).$$

So there are constants c_1, c_2, c_3 with

$A(t) = c_1, \quad B(t) = c_2, \quad C(t) = kc_1 t + c_3,$
and so $u(x,t) = \frac{c_1 x^2}{2} + c_2 x + kc_1 t + c_3$.
Since $u(x,0) = x^2$, we have

$x^2 = \frac{c_1 x^2}{2} + c_2 x + c_3 \Rightarrow c_1 = 2, \; c_2 = c_3 = 0$, so $u(x,t) = x^2 + 2kt$.

#10: a. We solve exercise #9 using the general formula:

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} y^2 \, dy$$

Let $p = \frac{x-y}{\sqrt{4kt}}$. Then $y = x - \sqrt{4kt}\,p$

and $u(x,t) = \frac{1}{\sqrt{\pi}} \int_{+\infty}^{-\infty} e^{-p^2} (x - \sqrt{4kt}\,p)^2 (-dp)$.

Cancel the minus sign by switching the limits of integration and expand and distribute the square to obtain

$$u(x,t) = \frac{1}{\sqrt{\pi}}\left[x^2 \int_{-\infty}^{\infty} e^{-p^2} dp - 2\sqrt{4kt}\, x \int_{-\infty}^{+\infty} p e^{-p^2} dp + 4kt \int_{-\infty}^{\infty} p^2 e^{-p^2} dp\right]$$

The first integral converges to $\sqrt{\pi}$. The second integral is zero since pe^{-p^2} is an odd function and the two integrals $\int_0^\infty$ and $\int_{-\infty}^0$ converge. So $u(x,t) = x^2 + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} dp$.

b. From exercise #9, we must have

$$x^2 + \frac{4\kappa t}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp = x^2 + 2\kappa t,$$

so $\displaystyle\int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp = \frac{\sqrt{\pi}}{2}$.

#16: We solve $u_t - \kappa u_{xx} + bu = 0$ for $-\infty < x < \infty$ with $u(x,0) = \phi(x)$, $b > 0$.

Set $v(x,t) = e^{bt} u(x,t)$. Then

$$v_t = be^{bt} u + e^{bt} u_t$$

$$v_{xx} = e^{bt} u_{xx}$$

$$v_t - \kappa v_{xx} = e^{bt}(bu + u_t - \kappa u_{xx}) = 0$$

$$v(x,0) = (1)\, u(x,0) = \phi(x).$$

Applying the solution formula gives

$$u(x,t) = e^{-bt} v(x,t)$$

$$= e^{-bt} \cdot \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\kappa t}} \phi(y)\, dy.$$

#18: We solve $u_t - \kappa u_{xx} + Vu_x = 0$ for $-\infty < x < \infty$, $u(x,0) = \phi(x)$.

Set $y = x - Vt$, so that $x = y + Vt$, and define $w(y,t) := u(y + Vt, t)$. Then

$$\cancel{w_t = u_x(y+Vt,t)\cdot\frac{\partial x}{\partial t} = u_x(y+Vt,y)\cdot V(}$$

$$\cancel{w_y = u_x(y+Vt,t), \quad w_{yy} = u_{xx}(y+Vt,t)}$$

$$W_t(y,t) = U_x(y+Vt,t)\frac{\partial x}{\partial t} + U_t(y+Vt,t)$$
$$= U_x(y+Vt,t)\cdot V + U_t(y+Vt,t).$$

$W_y = U_x(y+Vt)$, $W_{yy} = U_{xx}(y+Vt)$; so

$W_t - k W_{yy} = V U_x + U_t - k U_{xx} = 0$ using the PDE. Also,

$$w(y,0) = u(y+V(0),0)$$
$$= u(y,0)$$
$$= \phi(y).$$

So we can use the solution formula to obtain

$$w(y,t) = \frac{1}{\sqrt{4\pi k t}}\int_{-\infty}^{\infty} e^{\frac{-(y-z)^2}{4kt}}\phi(z)\,dz$$

Since $w(x-Vt,t) = u((x-Vt)+Vt,t)$
$= u(x,t)$ we have

$$u(x,t) = w((x-Vt),t)$$
$$= \frac{1}{\sqrt{4\pi k t}}\int_{-\infty}^{\infty} e^{-\frac{[(x-Vt)-z]^2}{4kt}}\phi(z)\,dz$$