HW #4 Solutions

Sec 2.5

#2 : a. Suppose $u(x,t) = f(x-at)$ solves $u_{tt} = c^2 u_{xx}$.
Computing the derivatives gives

$$a^2 f''(x-at) = u_{tt} = c^2 u_{xx} = c^2 f''(x-at)$$

Assuming f is not linear, we must have $f''(x-at) \neq 0$ at some point. Hence $a^2 = c^2$, so $a = \pm c$.

b Suppose $u(x,t) = f(x-at)$ solves $u_t = k u_{xx}$.
We have:

$(-a) f'(x-at) = f''(x-at)$. This is an ODE in $y = f'$; solving gives $f'(w) = A e^{-a \cdot w}$

$\Rightarrow f(w) = C e^{-aw} + B$, A, B, C arbitrary.

So $u(x,t) = f(x-at) = C e^{-a(x-at)} + B$. There are no restrictions on a, so the speed is arbitrary.

#3 : Let u satisfy $u_t = \frac{1}{2} u_{xx}$ and define

$$v(x,t) := t^{-1/2} e^{\frac{x^2}{2} t^{-1}} \underline{u}(xt^{-1}, t^{-1})$$

(Note the typo in the text here). Then $v_t = -\frac{1}{2} v_{xx}$ for $t > 0$.

After careful application of the product and chain rules, we find:

$$V_t = \left(-\tfrac{1}{2}\right)t^{-3/2}e^{\frac{x^2}{2}t^{-1}}u(xt^{-1},t^{-1}) + \left(-\tfrac{1}{2}\right)t^{-5/2}x^2e^{\frac{x^2}{2}t^{-1}}u(xt^{-1},t^{-1})$$
$$+ (-1)t^{-\frac{5}{2}}xe^{\frac{x^2}{2}t^{-1}}u_x(xt^{-1},t^{-1}) + (-1)t^{-5/2}e^{\frac{x^2}{2}t^{-1}}u_t(xt^{-1},t^{-1}).$$

and

$$V_{xx} = t^{-3/2}e^{\frac{x^2}{2}t^{-1}}u(xt^{-1},t^{-1}) + t^{-5/2}x^2e^{\frac{x^2}{2}t^{-1}}u(xt^{-1},t^{-1})$$
$$+ 2t^{-5/2}xe^{\frac{x^2}{2}t^{-1}}u_x(xt^{-1},t^{-1}) + t^{-5/2}e^{\frac{x^2}{2}t^{-1}}u_{xx}(xt^{-1},t^{-1}).$$

Using the PDE to replace $u_{xx}(xt^{-1},t^{-1})$ by $2u_t(xt^{-1},t^{-1})$ in the last term, it is then clear that $V_t = -\frac{1}{2}V_{xx}$ by comparing the expressions term by term.

$\boxed{\text{Sec 4.1}}$

#1: According to the discussion in the text, the string vibrates with a fundamental frequency of $\frac{\pi\sqrt{T}}{l\sqrt{\rho}}$. By clamping the string at its midpoint we cut the length in half, so the new fundamental frequency is $\frac{\pi\sqrt{T}}{(l/2)\sqrt{\rho}} = 2\left(\frac{\pi\sqrt{T}}{l\sqrt{\rho}}\right)$ (and in general, all the frequencies $\frac{\pi n\sqrt{T}}{l\sqrt{\rho}}$ are increased by a factor of 2). So the new note is one

octave higher than the original.
b: When the string is tightened, the tension T increases, so that the frequencies $\frac{n\pi\sqrt{T}}{\ell\sqrt{\rho}}$ increase, causing the note to rise.

#2: Putting the ends of the rod in the bath yields the boundry condition $u(0,t) = u(\ell,t) = 0$. The initial condition is $\phi(x) = u(x,0) \equiv 1$.
We are given that $1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{\ell} + \frac{1}{3}\sin\frac{3\pi x}{\ell} + \frac{1}{5}\sin\frac{5\pi x}{\ell} + \ldots\right)$
so $\phi(x) = \left(\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell}\right)$, where

$$A_{2k+1} = \frac{4}{\pi}(2k+1)^{-1}, \quad A_{2k} = 0 \quad \text{for } k = 0, 1, 2, 3, \ldots$$

Using the solution formula,

$$u(x,t) = \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin\left(\frac{(2n+1)\pi x}{\ell}\right) \cdot e^{-\left(\frac{n\pi}{\ell}\right)^2 kt}$$

#3: The potential outside the interval $(0,\ell)$ is infinite; this means the particle would require infinite energy to leave $(0,\ell)$, hence it is trapped in the interval. In particular, we must have $u(0,t) = u(\ell,t) = 0$. Let $u(x,t) = X(x)\,T(t)$.
Separating variables, we find that

$$-\frac{T'}{iT} = -\frac{X''}{X} = \lambda > 0.$$ We have already

seen that $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ and $X_n(x) = \sin\frac{n\pi x}{l}$.
Solving $T' = -i\lambda T$ we find
$T_n(t) = A_n e^{-i\lambda t} = A_n e^{-i\left(\frac{n\pi}{l}\right)^2 t}$, so the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-i\left(\frac{n\pi}{l}\right)^2 t} \sin\frac{n\pi x}{l}$$

#4: We solve:

$$u_{tt} = c^2 u_{xx} - r u_t \quad \text{for } 0 < x < l, \quad 0 < r < \frac{2\pi c}{l}$$

w/ I.C.'s $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$

and B.C.'s $u(0,t) = u(l,t) = 0$.

Separating variables yields

$$T''X = c^2 X'' T - r X T'$$

$$\Leftrightarrow \quad -\frac{T'' + rT'}{c^2 T} = -\frac{X''}{X} = \lambda.$$

We have already seen how to solve for X; we must have $X(x) = \sin\frac{n\pi x}{l}$ (or any linear combination of such functions). T satisfies

$$T'' + rT' + \lambda c^2 T = 0.$$

The characteristic equation for this ODE is

$$w^2 + rw + \lambda c^2 = 0 \quad \Rightarrow$$

$$w_n = \frac{-r}{2} \pm \frac{1}{2}\sqrt{r^2 - 4\left(\frac{\pi n}{l}\right)^2 c^2}$$

where we have substituted $\lambda = \left(\frac{\pi n}{l}\right)^2$.

Since $0 < r < \frac{2\pi c}{\ell}$, we have $r^2 < 4\left(\frac{\pi n}{\ell}\right)^2 c^2$ for all $n \geq 1$ so the roots w_n are complex. Given a root $w_n = \alpha_n + i\beta_n$, we have the solution

$$T(t) = A_n e^{\alpha_n t} \cos \beta_n t + B_n e^{\alpha_n t} \sin \beta_n t$$

where A_n, B_n are arbitrary. Hence the solution has form

$$u(x,t) = \sum_{n=1}^{\infty} e^{\alpha_n t}\left(A_n \cos \beta_n t + B_n \sin \beta_n t\right) \sin \frac{n\pi x}{\ell}$$

Note that

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} \quad \text{and}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \left[\alpha_n A_n + \beta_n B_n\right] \sin \frac{n\pi x}{\ell},$$

so if $\phi(x)$ and $\psi(x)$ are expressed as these sine series for some A_n and B_n, then our solution above also satisfies the initial conditions.