HW 5 Solutions

~~Pgs~~ $\boxed{\text{Sec } 4.2}$

#1: We solve $u_t = k u_{xx}$ on $0 < x < l$ w/ $u(0,t) = u_x(l,t) = 0$.

Assume $u(x,t) = X(x)\,T(t)$. Separating variables gives $\frac{T'}{kT} = \frac{X''}{X} = -\lambda$ for some constant λ. We must decide which values of λ are permissable, using the boundry conditions

$$0 = u(0,t) = X(0)\,T(t)$$
$$0 = u_x(l,t) = X'(l)\,T(t).$$

We may assume $T \neq 0$ (otherwise $u = 0$), so we have $X(0) = 0 = X'(l)$.

① Can $\lambda = 0$? Then the ODE for X becomes $X'' = 0 \Rightarrow X = Ax + B$ and $X' = A$. The boundry conditions give $0 = X(0) = B$, $0 = X'(l) = A$ so $X = 0$ and there are no eigen-functions.

② Now let λ be any nonzero complex number. Set $\mu = \sqrt{-\lambda}$;

the ODE becomes $X'' = \mu^2 X$ which has solution $X = Ae^{\mu x} + Be^{-\mu x}$ and hence

$X' = A\mu e^{\mu x} - B\mu e^{-\mu x}$. So

$0 = X(0) = A + B \Rightarrow -B = +A$

$0 = X'(l) = A\mu e^{\mu l} - B\mu e^{-\mu l}$
$= A\mu(e^{\mu l} + e^{-\mu l})$. We

may assume $A \neq 0$ (otherwise $X = 0$),

so $0 = e^{\mu l} + e^{-\mu l} \Rightarrow -1 = e^{2\mu l}$

$= e^{Re(2\mu l)} e^{i\, Im(2\mu l)} = e^{Re(2\mu l)}[\cos(Im\, 2\mu l) + i \sin(Im\, 2\mu l)]$

by Euler's formula. The second term lies on the unit circle in the complex plane, and the first term is a real number. If their product is equal to -1, we must have $e^{Re(2\mu l)} = 1$

$\Rightarrow Re(2\mu l) = 0$, and the angle $Im(2\mu l)$ must be of form $(2k+1)\pi$.

So $2\mu l = 0 + i\, Im(2\mu l) = i(2k+1)\pi$

for some integer k. Hence

$$\mu = \frac{i(k+\frac{1}{2})\pi}{l},$$ and

$$\lambda = -\mu^2 = \frac{(k+\frac{1}{2})^2\pi^2}{l^2},$$ where k is an integer ≥ 0. Given k, we find the eigenfunctions associated to λ_k:

$$X_k'' = -\lambda_k X_k \Rightarrow$$

$$X_k = A_k \cos\sqrt{\lambda_k}\, x + B_k \sin\sqrt{\lambda_k}\, x$$

and $$0 = X_k(0) = A_k$$

$$0 = X_k'(l) = B_k\sqrt{\lambda_k}\cos(\sqrt{\lambda_k}\cdot l)$$

$$= B_k\sqrt{\lambda_k}\cos\sqrt{\lambda_k}\, l$$

Since $\sqrt{\lambda_k} = \frac{(k+\frac{1}{2})\pi}{l}$, we have

$$\cos\sqrt{\lambda_k}\, l = \cos(k+\tfrac{1}{2})\pi = 0$$ so there is no condition on the choice of B_k;

So the eigenfunctions are, where $n \geq 0$

$$X_n(x) = B_n \sin\frac{(n+\frac{1}{2})\pi x}{l}.$$ The solutions to the ODE $\frac{T'}{kT} = -\lambda_n$ are

$$T_n(t) = A_n e^{-\lambda_n k t} = A_n e^{-\frac{(n+\frac{1}{2})^2\pi^2}{l^2} k t}$$

The general solution is then

$$u(x,t) = \sum_{n=0}^{\infty} C_n e^{-\frac{(n+\frac{1}{2})^2\pi^2}{l^2} k t} \sin \frac{(n+\frac{1}{2})\pi x}{l}$$

<u>#4</u>: We solve the diffusion equation $u_t = k u_{xx}$ on $-l < x < l$ with boundry conditions $u(-l,t) = u(l,t)$, $u_x(-l,t) = u_x(l,t)$.

~~a The eigenvalues are the numbers~~

~~a~~ As usual we separate variables: assume $u(x,t) = X(x)\,T(t)$; the PDE implies that

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda ,$$

and the boundry conditions imply that

$$X(-l) = X(l) \quad \text{and} \quad X'(-l) = X'(l)$$

<u>a.</u> The eigenvalues are the numbers λ for which there exists a nonzero function X satisfying $X'' = -\lambda X$ and the given boundry conditions

① Can $\lambda = 0$ be an eigenvalue?

Then we have $X'' = 0 \Rightarrow X = Ax + B$

The boundry conditions give $A(-l) + B = Al + B$

$\Rightarrow A = -A \Rightarrow A = 0$. Also
$X' = A$ so certainly

$X'(-l) = X'(l)$. So $\lambda = 0$ is an eigenvalue, with corresponding eigenfunctions being the constant functions.

② Now let λ be any nonzero complex number. Then ~~we~~ set $\mu = \sqrt{-\lambda}$ so that $X'' = -\lambda X = \mu^2 X$ has solution $X = Ae^{\mu t} + Be^{-\mu t}$. The first boundry condition implies that

$$Ae^{-\mu l} + Be^{\mu l} = Ae^{\mu l} + Be^{-\mu l}$$

Multiplying by $e^{\mu l}$ and rearranging gives

$$(B-A)e^{2\mu l} = B - A.$$ If $B - A \neq 0$ then $e^{2\mu l} = 1$ from which it follows that $Re(2\mu l) = 0$ and $Im(2\mu l) = 2\pi n$ for some ~~[illegible]~~ integer n (see #1 above for a similar but more detailed calculation).

So $2\mu l = i2\pi n$ for some nonzero integer n ($n \neq 0$ b/c otherwise we have $\mu = 0 \Rightarrow \lambda = 0$, but we assumed $\lambda \neq 0$),

$$\Rightarrow \lambda = -\mu^2 = \frac{\pi^2 n^2}{l^2}, \quad n = 1, 2, 3, \ldots$$

On the other hand, if $A = B$ we must apply the second boundry condition. In this case,

$$\bar{X}(x) = Ae^{\mu x} + Ae^{-\mu x}, \Rightarrow$$

$$\bar{X}'(x) = A\mu(e^{\mu x} - e^{-\mu x}).$$ The BC implies that

$$A\mu(e^{-\mu l} - e^{\mu l}) = A\mu(e^{\mu l} - e^{-\mu l})$$

Cancel, rearrange, and multiply by $e^{2\mu l}$ to obtain $e^{2\mu l} = 1$. We just saw above that this condition leads to $\lambda = \frac{n^2\pi^2}{l^2}$ for some $n > 0$.

Combining 1 and 2, we find that the eigenvalues are $\lambda_n = \frac{n^2\pi^2}{l^2}$, $n = 0, 1, 2, \ldots$

b We know the eigenvalues from part a; we now solve the ODE's $\frac{T'}{kT} = \frac{\bar{X}''}{\bar{X}} = -\lambda_n$. When $\lambda = 0$, we saw that the eigenfunctions were constants: $\bar{X}_0(x) = A_0$. If $\lambda_n = \frac{n^2\pi^2}{l^2}$ for $n > 0$, then $\bar{X}'' = -\lambda_n \bar{X}$ has the solution

$$X_n = A_n \sin \frac{n\pi x}{l} + B_n \cos \frac{n\pi x}{l}$$

Since X_n and X_n' have period $2l$ (and smaller as n increases), we also have $X_n(-l) = X_n(l)$ and $X'_n(-l) = X_n'(l)$.

For the ODE $T' = -\lambda k T$, if $\lambda = 0$ then $T' =$ constant, and if $\lambda = \frac{n^2\pi^2}{l^2}$ then $T = A_n e^{-\frac{n^2\pi^2}{l^2} k t}$ (in fact, the latter result still holds for $n = 0$).

So the general solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \left[\left(A_n \cancel{\cos} \sin \frac{n\pi x}{l} + B_n \cos \frac{n\pi x}{l} \right) \cdot e^{-\frac{n^2\pi^2}{l^2} k t} \right].$$

Sec 5.1

#2: Let $\phi(x) = x^2$ for $0 \le x \le 1 = l$.

a. Fourier sine series: $x^2 = \sum_{n=1}^{\infty} A_n \sin n\pi x$

where $A_n = \frac{2}{1} \int_0^1 x^2 \sin n\pi x \, dx$

$$= 2\left[\frac{-1}{n\pi} x^2 \cos n\pi x \Big|_0^1 + \frac{2}{n\pi} \int_0^1 x \cos n\pi x \, dx \right]$$

$$= \frac{2}{n\pi} (-1)^{n+1} + \frac{2^2}{n\pi} \left[\frac{1}{n\pi} x \sin n\pi x \Big|_0^1 - \frac{1}{n\pi} \int_0^1 \sin n\pi x \, dx \right]$$

$$= \frac{2}{n\pi}(-1)^{n+1} + \frac{2^2}{(n\pi)^3}\cos n\pi x \Big|_0^1$$

$$= \frac{2}{n\pi}(-1)^{n+1} + \frac{2^2}{(n\pi)^3}\left((-1)^n - 1\right)$$

b: Cosine series: $x^2 = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos n\pi x$

where $B_n = 2\int_0^1 x^2 \cos n\pi x \, dx$

$$= 2\left[\frac{1}{n\pi} x^2 \sin n\pi x \Big|_0^1 - \frac{2}{n\pi}\int_0^1 x \sin n\pi x \, dx\right]$$

$$= -\frac{2^2}{n\pi}\left[-\frac{1}{n\pi} x \cos n\pi x \Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x \, dx\right]$$

$$= \frac{2^2}{(n\pi)^2}(-1)^n - \frac{2^2}{(n\pi)^2}\cdot\frac{1}{n\pi}\sin n\pi x \Big|_0^1$$

$$= \frac{2^2}{n^2\pi^2}(-1)^n \quad \text{for } n > 0, \text{ and}$$

$$B_0 = 2\int_0^1 x^2 \, dx = \frac{2}{3}.$$

#3: Let $\phi(x) = x$ be given on $(0, l)$

by $x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \dots\right)$

$$= \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi x}{l}$$

a. We integrate term by term to obtain

$$\frac{1}{2}x^2 = \frac{2l}{\pi}\left(-\frac{l}{\pi}\cos\frac{\pi x}{l} + \frac{l}{2\pi}\cos\frac{2\pi x}{l} - \frac{l}{3\pi}\cos\frac{3\pi x}{l} + \dots\right)$$

$$+ C = C + \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos\frac{n\pi x}{l}$$

Comparing this to the formula for Fourier cosine series, we must have $C = \frac{1}{2}A_0 = \frac{1}{2}\left(\frac{2}{l}\int_0^l \frac{x^2}{2}\cos(0)\,dx\right)$

$$= \frac{1}{6l}x^3\Big]_0^l = \frac{l^2}{6},$$ so

$$\frac{1}{2}x^2 = \frac{l^2}{6} + \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos\frac{n\pi x}{l}.$$

b: Plugging in $x = 0$ gives

$$0 = \frac{l^2}{6} + \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\frac{l^2}{6}\cdot\frac{\pi^2}{2l^2} = -\frac{\pi^2}{12}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \frac{\pi^2}{12}.$$

#8: We solve $u_t = u_{xx}$ on $0 \le x \le 1$ with B.C.s $u(0,t) = 0$, $u(1,t) = 1$, and I.C. $u(x,0) = \phi(x) = \begin{cases} \frac{5x}{2} & 0 < x < \frac{2}{3} \\ 3 - 2x & \frac{2}{3} < x < 1 \end{cases}$

We first find the equilibrium solution $\mathcal{U}(x)$, which is the time-independent solution to $u_t = u_{xx}$ satisfying the same BC's as above: $\mathcal{U}(0) = 0$, $\mathcal{U}(1) = 1$. Since $\mathcal{U}$ does not depend on t, we have $\mathcal{U}_{xx} = \mathcal{U}_t = 0 \Rightarrow \mathcal{U}(x) = Ax + B$.

The BC's then imply that $0 = B$ and $1 = A(1) = A$, so $\mathcal{U}(x) = x$.

Now consider the problem

$$\left.\begin{array}{l} v_t = v_{xx} \quad \text{for } 0 < x < 1, \\ v(0,t) = v(1,t) = 0, \quad v(x,0) = \hat{\phi}(x) \end{array}\right\} (*)$$

where $\hat{\phi}(x) = \phi(x) - \mathcal{U}(x)$

<u>Claim</u>: If v is the solution of $(*)$ then $u(x,t) = v(x,t) + \mathcal{U}(x)$ is the solution to our original problem.

<u>Pf</u>: Since v and $\mathcal{U}$ both satisfy the diffusion equation $u_t = u_{xx}$, so does the sum $u = v + \mathcal{U}$. At the boundry we have $u(0,t) = v(0,t) + \mathcal{U}(0) = 0 + 0$ and $u(1,t) = v(1,t) + \mathcal{U}(1) = 0 + 1$. At time $t = 0$ we have $u(x,0) = v(x,0) + \mathcal{U}(x)$
$= (\phi(x) - \mathcal{U}(x)) + \mathcal{U}(x) = \phi(x)$.

So all that remains is to find the solution v of $(*)$. We have

$$v(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \cdot A_n \sin n\pi x \quad \text{where}$$

$$A_n = 2\int_0^1 \hat{\phi}(x) \sin n\pi x \, dx. \quad \text{We have}$$

$$\hat{\phi}(x) = \phi(x) - x = \begin{cases} \frac{3x}{2} & 0 < x < \frac{2}{3} \\ 3 - 3x & \frac{2}{3} < x < 1, \end{cases}$$

$$\text{so } A_n = \underbrace{3\int_0^{2/3} x \sin n\pi x \, dx}_{I_1} + \underbrace{6\int_{2/3}^1 \sin n\pi x \, dx}_{I_2} - \underbrace{6\int_{2/3}^1 x \sin n\pi x \, dx}_{I_3}$$

$$I_1 = 3\left[\frac{-x}{n\pi} \cos n\pi x \Big|_0^{2/3} + \frac{1}{n\pi}\int_0^{2/3} \cos n\pi x \, dx \right]$$

$$= \frac{-2}{n\pi}\cos\left(\frac{2n\pi}{3}\right) + \frac{3}{n^2\pi^2} \sin n\pi x \Big|_0^{2/3}$$

$$= \frac{-2}{n\pi}\cos\left(\frac{2n\pi}{3}\right) + \frac{3}{n^2\pi^2} \sin \frac{2n\pi}{3}$$

$$I_2 = 6\left[-\frac{1}{n\pi} \cos n\pi x \Big|_{2/3}^1 \right] = \frac{-6}{n\pi}\left((-1)^n - \cos \frac{2n\pi}{3} \right)$$

$$= \frac{6}{n\pi} \cos \frac{2n\pi}{3} + \frac{6(-1)^{n+1}}{n\pi}$$

$$I_3 = 6\left[-\frac{x}{n\pi} \cos n\pi x \Big|_{2/3}^1 + \frac{1}{n\pi}\int_{2/3}^1 \cos n\pi x \, dx \right]$$

$$= \frac{-6}{n\pi}\left((-1)^n - \frac{2}{3}\cos \frac{2n\pi}{3} \right) + \frac{6}{n^2\pi^2} \sin n\pi x \Big|_{2/3}^1$$

$$= \frac{6}{n\pi}(-1)^{n+1} + \frac{4}{n\pi}\cos\frac{2n\pi}{3} - \frac{6}{n^2\pi^2}\sin\frac{2n\pi}{3}$$

So $A_n = I_1 + I_2 - I_3$

$$= \frac{9}{n^2\pi^2} \sin \frac{2n\pi}{3}$$

We can use the sine series for x to write down the solution $u = v + x$ as a single series:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \cdot A_n \cdot \sin n\pi x + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \sin n\pi x$$

$$= \sum_{n=1}^{\infty} \left[\frac{9}{n^2\pi^2} \sin\left(\frac{2n\pi}{3}\right) + \frac{2(-1)^{n+1}}{\pi n} \right] \sin n\pi x \, e^{-n^2\pi^2 t}$$

$\boxed{\text{Sec } 5.2}$

#2: The period of $\cos x$ is 2π and the period of $\cos \alpha x$ is $\frac{2\pi}{\alpha}$. Then any number that is a multiple of both 2π and $\frac{2\pi}{\alpha}$ will be a period for the sum $\cos x + \cos \alpha x$. In particular, the smallest period is the least common multiple $\mathrm{lcm}(2\pi, \frac{2\pi}{\alpha})$. To make this more explicit, write $\alpha = \frac{p}{q}$ where p and q have no common prime factors. Then the least common multiple of 2π and $\frac{2\pi}{\alpha} = \frac{2\pi q}{p}$ is the first multiple $k \cdot \frac{2\pi q}{p}$ that is also a multiple of 2π. If p and q share no common factors, then the first time this happens is when $k = p$, so the period is $2\pi q$.

#4: a. The coefficients of cosine in the Fourier series for $\phi(x)$ are given by $A_n = \frac{1}{l}\int_{-l}^{l} \phi(x) \cos \frac{n\pi x}{l}\, dx$.

Since ϕ is assumed odd, $\phi(x) \cos \frac{n\pi x}{l}$ is odd, so these integrals are all 0.

b. As in part a., the integrand in the expressions for the coefficients of sine in the Fourier series is odd, so the integrals are all zero.

#8: a. Suppose ϕ is even, i.e. $\phi(-x) = \phi(x)$ for all x. Then by definition,

$$\phi'(-x) = \lim_{h \to 0} \frac{\phi(-x+h) - \phi(-x)}{h} = \lim_{h \to 0} \frac{\phi(x+(-h)) - \phi(x)}{h}$$

$$= \cancel{\lim} \ (-1)\left[\lim_{h\to 0} \frac{\phi(x+(-h)) - \phi(x)}{(-h)}\right]$$

Let $k = -h$ and note that as $h \to 0$ we have $k = -h \to 0$. So the limit becomes
$$(-1)\left[\lim_{k\to 0} \frac{\phi(x+k) - \phi(x)}{k}\right] = -\phi'(x).$$

Hence ϕ' is odd. The case for ϕ odd is similar.

<u>b.</u> Again, assume ϕ is even, and define $F(x) := \int_0^x \phi(t)\,dt$. Then $F(-x) = \int_0^{-x} \phi(t)\,dt$. Make the substitution $u = -t$, $du = -dt$. Then $u(0) = 0$ and $u(-x) = x$, and we have

$$F(-x) = \int_0^x \phi(-u)(-du)$$

$$= -\int_0^x \phi(-u)\,du$$

using that ϕ is even $\;= -\int_0^x \phi(u)\,du = -F(x)$;

so F is odd. The case for ϕ odd is similar.

#11: We compute the Fourier series of e^x on $(-l, l)$:

~~Real form~~ Complex form: $e^x = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$

where $c_n = \frac{1}{2l}\int_{-l}^{l} e^x e^{-in\pi x/l}\,dx$

$= \frac{1}{2l}\int_{-l}^{l} e^{x(1-\frac{in\pi}{l})}\,dx = \frac{1}{2l}\frac{1}{(1-\frac{in\pi}{l})} e^{x(1-\frac{in\pi}{l})}\Big|_{-l}^{l}$

$= \frac{1}{2l}\frac{1}{(1-\frac{in\pi}{l})}\left(e^{l-in\pi} - e^{-l+in\pi}\right)$

$= \frac{1}{l-in\pi}\,\frac{1}{2}\left(e^{-in\pi}e^{l} - e^{in\pi}e^{-l}\right)$

Note that $e^{-in\pi} = e^{in\pi} = \cos n\pi = (-1)^n$

(you can show this using Euler's formula),

so we have

$= \frac{(-1)^n}{l-in\pi}\sinh(l)$. Also,

$\frac{1}{l-in\pi} = \frac{l+in\pi}{(l-in\pi)(l+in\pi)} = \frac{l+in\pi}{l^2+n^2\pi^2}$, so the coefficients become $(-1)^n\frac{l+in\pi}{l^2+n^2\pi^2}\sinh(l)$

and $e^x = \sum_{n=-\infty}^{\infty}(-1)^n\cdot\frac{l+in\pi}{l^2+n^2\pi^2}\sinh(l)\,e^{in\pi x/l}$

Real Form: $e^x = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}$.

We can compute the coefficients A_n and B_n directly using the formula, but we can also recover them from the complex form; we break off the $n=0$ term and pair the terms for n and $-n$ together:

$$e^x = \frac{\sinh(\ell)}{\ell} + \sum_{n=1}^{\infty} (-1)^n \frac{\ell + in\pi}{\ell^2+n^2\pi^2} \sinh(\ell)\, e^{in\pi x/\ell}$$

$$+ \sum_{n=1}^{\infty} (-1)^{(-n)} \frac{\ell + i(-n)\pi}{\ell^2+(-n)^2\pi^2} \sinh(\ell)\, e^{i(-n)\pi x/\ell}$$

$$= \frac{\sinh(\ell)}{\ell} + \sum_{n=1}^{\infty} (-1)^n \sinh(\ell) \left[\frac{\ell + in\pi}{\ell^2+n^2\pi^2} e^{in\pi x/\ell} + \frac{\ell - in\pi}{\ell^2+n^2\pi^2} e^{-in\pi x/\ell} \right]$$

The term in square brackets is

$$\frac{\ell}{\ell^2+n^2\pi^2}\left(e^{in\pi x/\ell} + e^{-in\pi x/\ell}\right) + \frac{in\pi}{\ell^2+n^2\pi^2}\left(e^{in\pi x/\ell} - e^{-in\pi x/\ell}\right)$$

$$= \frac{(2)\ell}{\ell^2+n^2\pi^2} \cos\frac{n\pi x}{\ell} + \frac{(2i)\,in\pi}{\ell^2+n^2\pi^2} \sin\frac{n\pi x}{\ell}$$

Plugging this in, we get the expansion

$$e^x = \frac{\sinh(\ell)}{\ell} +$$

$$\sum_{n=1}^{\infty}\left(\underbrace{\frac{(-1)^n \sinh(\ell)\cdot 2\ell}{\ell^2+n^2\pi^2}}_{A_n} \cos\frac{n\pi x}{\ell} + \underbrace{\frac{(-1)^{n+1} \sinh(\ell)\cdot 2n\pi}{\ell^2+n^2\pi^2}}_{B_n} \sin\frac{n\pi x}{\ell} \right)$$