HW #6 Solutions

Sec 5.3

#1: a. The vectors (a,b,c) orthogonal to both $(1,1,1)$ and $(1,-1,0)$ must have $a\cdot 1 + b\cdot 1 + c\cdot 1 = 0$ and $a\cdot 1 + b\cdot(-1) + c(0) = 0$. The second equation $\Rightarrow$ $b = a$, and first then gives $c = -2a$, so the vectors have form $(a,b,c) = (a, a, -2a) = a(1,1,-2)$ for all real numbers a. We also could have computed the cross product $(1,1,1) \times (1,-1,0)$.

b. Set $v_1 = (1,1,1)$, $v_2 = (1,-1,0)$, $v_3 = (1,1,-2)$, $c = (2,-3,5)$. Since $\{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathbb{R}^3$, we have

$$c = \frac{c\cdot v_1}{v_1\cdot v_1}v_1 + \frac{c\cdot v_2}{v_2\cdot v_2}v_2 + \frac{c\cdot v_3}{v_3\cdot v_3}$$

$$= \frac{4}{3}v_1 + \frac{5}{2}v_2 - \frac{11}{6}v_3$$

#2: a. For any constant c, we have $(x,c) = \int_{-1}^{1} x\cdot c\, dx = c\int_{-1}^{1} x\, dx = 0$, so x and c are orthogonal.

b. Write $Q(x) = Ax^2 + Bx + C$. We have

$$(Q,1) = \int_{-1}^{1} Q\cdot 1\, dx = \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\Big|_{-1}^{1} = \frac{2A}{3} + 2C$$

$$(Q, x) = \int_{-1}^{1} Ax^3 + Bx^2 + Cx \, dx$$
$$= \frac{A}{4}x^4 + \frac{B}{3}x^3 + \frac{C}{2}x^2 \Big|_{-1}^{1} = \frac{2B}{3}$$

So $(Q, x) = 0 \Rightarrow B = 0$, and $(Q, 1) = 0$ $\Rightarrow \frac{2A}{3} + 2C = 0 \Rightarrow C = -\frac{A}{3}$. So any quadratic of form $Q(x) = Ax^2 - \frac{A}{3}$ is orthogonal to both 1 and x.

c. Now let $Q(x) = Ax^2 + Bx + C$ be an arbitrary quadratic and $G(x) = ax^3 + bx^2 + cx + d$ a cubic. Distributing $(G)(Q)$ gives

$$GQ = \overbrace{(aA)}^{p_5}x^5 + \overbrace{(aB + bA)}^{p_4}x^4 + \overbrace{(aC + bB + cA)}^{p_3}x^3$$
$$+ \underbrace{(bC + cB + dA)}_{p_2}x^2 + \underbrace{(cC + dB)}_{p_1}x + \underbrace{(dC)}_{p_0}$$

Then G orthogonal to Q means that

$$0 = \int_{-1}^{1} GQ \, dx = \frac{p_5}{6}x^6 + \frac{p_4}{5}x^5 + \frac{p_3}{4}x^4 + \frac{p_2}{3}x^3 + \frac{p_1}{2}x^2 + p_0 x \Big|_{-1}^{1}$$
$$= \frac{2}{5}p_4 + \frac{2}{3}p_2 + 2p_0$$

$$= \left(\tfrac{2}{5}B\right)a + \left(\tfrac{2}{5}A + \tfrac{2}{3}C\right)b + \left(\tfrac{2}{3}B\right)c + \left(\tfrac{2}{3}A + 2C\right)d$$

If this holds for all A, B, C then we must have $0 = \left(\tfrac{2}{5}B\right)a + \left(\tfrac{2}{3}B\right)c$ and

$$0 = \left(\tfrac{2}{5}A + \tfrac{2}{3}C\right)b + \left(\tfrac{2}{3}A + 2C\right)d.$$

The first equation gives $c = -\frac{3}{5}a$, and the second can hold for all A, C only if $b = d = 0$. So the cubic functions orthogonal to all quadratics are those of form

$$G(x) = ax^3 - \frac{3a}{5}x$$

#3: We solve $u_{tt} = c^2 u_{xx}$ for $0 < x < l$ with BC's $u(0,t) = 0 = u_x(l,t)$ and IC's $u(x,0) = x$, $u_t(x,0) = 0$.

Separate variables: $u(x,t) = X(x)\,T(t)$. The PDE gives $\frac{T''}{-c^2 T} = \frac{X''}{-X} = \lambda$.

Notice that the BC's are symmetric: for any functions f and g both satisfying these BC's, we have

$$f'(x)g(x) - f(x)g'(x)\Big|_{x=0}^{x=l} = \big(f'(l)g(l) - f(l)g'(l)\big) - \big(f'(0)g(0) - f(0)g'(0)\big) = 0$$

since at least one term in each summand is zero. According to Thm 5.3.2, all eigenvalues λ are real. Moreover, $f(x)f'(x)\Big|_0^l = 0 \le 0$, so by

Thm 5.3.3, all eigenvalues are nonnegative.

Can $\lambda = 0$? In this case we have the ODE $X'' = 0 \Rightarrow X = Ax + B$, and the BC's $\Rightarrow$ $0 = X(0) = B$ and $0 = X'(l) = A$, so $X = 0$ is the only solution; hence 0 is not an eigenvalue.

So all eigenvalues are positive. Write $\beta = \sqrt{\lambda}$. Then $X'' = -\lambda X = -\beta^2 X$ has solutions $X(x) = A\cos\beta x + B\sin\beta x$. The BC's give $0 = X(0) = A(1) + B(0) = A$ and $0 = X'(l) = 0 + B\beta\cos\beta l$. We must have B and $\beta \neq 0$, therefore

$\beta l = \underline{(n+\frac{1}{2})\pi}$ for some integer $n \geq 0$

So $\lambda = \beta^2 = \dfrac{(n+\frac{1}{2})^2\pi^2}{l^2}$ and the corresponding eigenfunctions are $X(x) = A\sin\frac{(n+\frac{1}{2})\pi}{l}x$.

Similarly, the ODE $\dfrac{T''}{-Tc^2} = \lambda$ has solutions

$$T(t) = A\cos c\cdot\frac{(n+\frac{1}{2})\pi}{l}t + B\sin c\frac{(n+\frac{1}{2})\pi}{l}t$$

$$\text{So } u(x,t) = \sum_{n=0}^{\infty}\left(A_n\cos\frac{c(n+\frac{1}{2})\pi}{l}t + B_n\sin\frac{c(n+\frac{1}{2})\pi}{l}t\right)\sin\frac{(n+\frac{1}{2})\pi}{l}x$$

We use the ICs to determine the coefficients A_n and B_n. We have $u(x,0) = \sum_{n=0}^{\infty} A_n \sin \frac{(n+\frac{1}{2})\pi}{l} x$; the first initial condition is $u(x,0) = x$, so we must find the series expansion of this function in terms of the functions $\sin \frac{(n+\frac{1}{2})\pi}{l} x$.

I.e. we write $x = \sum_{n=0}^{\infty} A_n \sin \frac{(n+\frac{1}{2})\pi}{l} x$; according to Thm 5.3.1, the coefficients are given by

$$A_n = \frac{\left(x, \sin \frac{(n+\frac{1}{2})\pi x}{l}\right)}{\left(\sin \frac{(n+\frac{1}{2})\pi x}{l}, \sin \frac{(n+\frac{1}{2})\pi}{l} x\right)}$$

The terms in the denominator are equal to $\frac{l}{2}$ for all n (Hint: to compute $\int \sin^2 x\,dx$, write $\sin^2 x = \sin x \cdot \sin x$ and integrate by parts. In the resulting integral, replace $\cos^2 x$ by $1 - \sin^2 x$ and solve the equation for $\int \sin^2 x\,dx$).

So $$A_n = \frac{2}{l} \int_0^l x \sin \frac{(n+\frac{1}{2})\pi x}{l}\,dx$$

$$= \dots = \frac{2l}{(n+\frac{1}{2})^2 \pi^2} (-1)^n.$$

Differentiating $u(x,t)$ gives

$u_t(x,0) = \sum_{n=0}^{\infty} B_n \cdot \frac{c(n+\frac{1}{2})\pi}{l} \sin \frac{(n+\frac{1}{2})\pi}{l} x$. Since

The second IC is $u_t(x,0) = 0$, we may take $B_n = 0$ for all n.

So the solution is

$$u(x,t) = \sum_{n=0}^{\infty} \frac{2\ell\,(-1)^n}{(n+\frac{1}{2})^2\pi^2} \cos\left(\frac{c(n+\frac{1}{2})\pi}{\ell}t\right) \sin\left(\frac{(n+\frac{1}{2})\pi}{\ell}x\right)$$

#6: $X' = \lambda X$ has solutions $X = Ae^{\lambda x}$.

The BCs imply that

$$A = X(0) = X(1) = Ae^{\lambda}, \text{ so}$$

$1 = e^{\lambda}$. We can use Euler's formula to show that $\operatorname{Re}\lambda = 0$ and $\operatorname{Im}\lambda = 2\pi n$, so the eigenvalues are $\boxed{\lambda = 2\pi n i.}$ If $m \neq n$, we have

$$(X_m, X_n) = \int_0^1 e^{2\pi m i x} e^{2\pi n i x}\,dx = \int_0^1 e^{2\pi(m+n)ix}\,dx$$

$$= \frac{1}{2\pi(m+n)i} e^{2\pi(m+n)ix}\Big|_0^1 = \frac{1}{2\pi(m+n)i}\left(e^{2\pi(m+n)i} - 1\right)$$

But for any integers m and n, $e^{2\pi(m+n)i} = 1$, (use Euler's formula to see this) so the eigenfunctions are orthogonal.

#9: Suppose f and g satisfy the BC's
$X(b) = \alpha X(a) + \beta X'(a)$, $X'(b) = \gamma X(a) + \delta X'(a)$

Then $f'(x)g(x) - f(x)g'(x) \Big|_a^b$

$= f'(b)g(b) - f(b)g'(b) - f'(a)g(a) + f(a)g'(a)$

$= [\gamma f(a) + \delta f'(a)][\alpha g(a) + \beta g'(a)] - [\alpha f(a) + \beta f'(a)][\gamma g(a) + \delta g'(a)]$
$- f'(a)g(a) + f(a)g'(a)$

$= \gamma\alpha f(a)g(a) + \delta\alpha f'(a)g(a) + \gamma\beta f(a)g'(a) + \delta\beta f'(a)g'(a)$
$- \alpha\gamma f(a)g(a) - \beta\gamma f'(a)g(a) - \alpha\delta f(a)g'(a) - \beta\delta f'(a)g'(a)$
$- f'(a)g(a) + f(a)g'(a)$

$= [(\alpha\delta - \beta\gamma) - 1] f'(a)g(a) + [(-\alpha\delta + \beta\gamma) + 1] f(a)g'(a)$

$= [(\alpha\delta - \beta\gamma) - 1](f'(a)g(a) - f(a)g'(a))$

One implication is clear: If $\alpha\delta - \beta\gamma = \cancel{0}\,1$, then $f'(x)g(x) - f(x)g'(x) \Big|_a^b = 0$, so the BC's are symmetric.

For the other direction we assume that the BC's are symmetric, i.e. for any f, g satisfying the BC's, we have $f'(x)g(x) - f(x)g'(x)\Big|_a^b = 0$. Given such f and g, the above computation shows that $[(\alpha\delta - \beta\gamma) - 1](f'(a)g(a) - f(a)g'(a)) = 0$. If the second term is nonzero, we can cancel it to get $(\alpha\delta - \beta\gamma) - 1 = 0$, or $\alpha\delta - \beta\gamma = 1$. So we are done if we can

~~Should~~ show that there are functions f and g that satisfy the BC's and are such that $f'(a)g(a) - f(a)g'(a) \neq 0$. One (rather involved) way to do this is to find ~~[illegible]~~ cubic functions f and g that satisfy the BC's and have $f'(a) = g(a) = 1$, $f(a) = g'(a) = 0$. For such f and g we then have $f'(a)g(a) - f(a)g'(a) = 1 \neq 0$, as desired. The existence of such f + g can be verified in many ways; below we use linear algebra to guarantee the existence of the cubic $g(x)$ with the right properties:

Assume $g(x) = Ax^3 + Bx^2 + Cx + D$, and that $g(a) = 1$, $g'(a) = 0$. Plugging these values into the boundary conditions gives $g(b) = \alpha$, $g'(b) = \gamma$. Using the expression for g we have the system of equations

$$
\begin{aligned}
g(a) &= Aa^3 + Ba^2 + Ca + D = 1 \\
g(b) &= Ab^3 + Bb^2 + Cb + D = \alpha \\
g'(a) &= 3Aa^2 + 2Ba + C + 0 = 0 \\
g'(b) &= 3Ab^2 + 2Bb + C + 0 = \gamma .
\end{aligned}
$$

In matrix notation this ~~[illegible]~~ becomes

$$\begin{pmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ 3a^2 & 2a & 1 & 0 \\ 3b^2 & 2b & 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \\ 0 \\ \gamma \end{pmatrix}$$

The matrix has determinant $-(b-a)^4 \neq 0$, so there <u>is</u> a solution, i.e. a set of constants A, B, C, and D such that $g(x)$ has the desired properties. In fact this also proves the existence of $f(x)$, since the only change would occur in the right-hand side of the equation.

$\boxed{\text{Sec } 5.4}$ **#2**: Let $\sum_{n=1}^{\infty} f_n$ converge uniformly to f on $[a,b]$, and define $M_N := \max_{a \le x \le b} |f(x) - \sum_{n=1}^{N} f_n(x)|$. Then uniform convergence just means that $M_N \to 0$ as $N \to \infty$. Given a point $a \le x_0 < b$ we certainly have $|f(x_0) - \sum_{n=1}^{N} f_n(x_0)| \le M_N$ so uniform convergence implies that $|f(x_0) - \sum_{n=1}^{N} f_n(x_0)| \to 0$ as $N \to \infty$. Since x_0 was an arbitrary point, $\sum_{n=1}^{\infty} f_n$ converges pointwise to f on the interval (a,b).

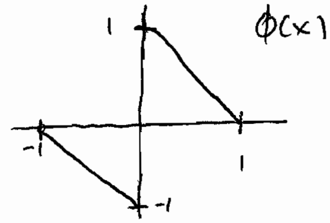
Similarly, for any x in the interval $[a,b]$ we have $|f(x) - \sum_{n=1}^{N} f_n(x)|^2 \le M_N^2$, so $0 \le \int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \le \int_a^b M_N^2 dx$

$= M_N^2 \cdot (b-a) \to 0$ as $N \to 0$, hence $\sum_{n=1}^{\infty} f_n$ converges to f in the L^2 sense.

#4: For $n > 1$, g_n is zero outside the interval $[0,1]$, so we will check for L^2 convergence on this interval. Since g_n is just a step function, we have

$$\int_0^1 |g_n(x) - 0|^2 dx = \int_0^1 g_n(x)\, dx = \begin{cases} \text{length } [\frac{1}{4} - \frac{1}{n^2}, \frac{1}{4} + \frac{1}{n^2}) & n \text{ odd} \\ \text{length } [\frac{3}{4} - \frac{1}{n^2}, \frac{3}{4} + \frac{1}{n^2}) & n \text{ even} \end{cases}$$

$\Big| = \dfrac{2}{n^2}$ for all $n \geq 1$. This last quantity $\to 0$ as $n \to \infty$, so $g_n \to 0$ in the L^2 sense. On the other hand, g_n does not converge pointwise to 0 on the interval $(0,1)$: for every odd n we have $|g_n(\frac{1}{4}) - 0| = 1$, so $|g_n(\frac{1}{4}) - 0|$ cannot possibly converge to zero as $n \to \infty$.



#7: Let $\phi(x) = \begin{cases} -1-x & -1 < x < 0 \\ 1-x & 0 < x < 1 \end{cases}$

a. The Fourier series for ϕ on $(-1,1)$ is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x + B_n \sin n\pi x$$

where $A_n = \int_{-l}^{l} \phi(x) \cos n\pi x \, dx = 0$ since $\phi(x) \cdot \cos n\pi x$ is odd, and

$$B_n = \int_{-1}^{1} \phi(x) \sin n\pi x \, dx$$

$$= -\int_{-1}^{0} (1+x) \sin n\pi x \, dx + \int_{0}^{1} (1-x) \sin n\pi x \, dx$$

$$= -\int_{-1}^{0} \sin n\pi x \, dx - \int_{-1}^{0} x \sin n\pi x \, dx + \int_{0}^{1} \sin n\pi x \, dx - \int_{0}^{1} x \sin n\pi x \, dx$$

$$= 2\left(\int_0^1 \sin n\pi x \, dx - \int_0^1 x \sin n\pi x \, dx \right)$$

$$= 2\left(\frac{1}{n\pi}\left((-1)^{n+1} + 1\right) - \frac{1}{n\pi}(-1)^{n+1} \right)$$

$$= \frac{2}{n\pi}.$$

b: $\phi(x) = \frac{2}{\pi}\sin \pi x + \frac{1}{\pi}\sin 2\pi x + \frac{2}{3\pi}\sin 3\pi x + \ldots$

c: $\int_{-1}^1 |\phi(x)|^2 dx = 2\int_0^1 x^2 dx = \frac{2}{3} < \infty$

so by Thm 5.4.3 the series converges in the L^2 sense.

d: Since ~~ϕ is p and $\phi' = \{$~~

ϕ and ϕ' are piecewise continuous, the series converges pointwise (although not to $\phi(x)$) by Thm 5.4.4 b.

e: ϕ is not continuous on $(-1,1)$, so we cannot conclude that the series converges uniformly.

#8: We use the theorems of section 5.4:

a. $f(x) = x^3$ on $(0, l)$.

i: $\int_0^l |x|^3 dx = \frac{l^4}{4} < \infty$ so have L^2 convergence by Thm 3.

ii: f and f' are CTS $\Rightarrow$ pointwise convergence by Thm 4.

~~b) f(x)~~

iii: The eigenfunctions $\sin n\pi x$ arise from Dirichlet boundry conditions, but $f(l) = l^3 \neq 0$, so f does not meet the conditions of Thm 2 and we cannot conclude uniform convergence.

b: $f(x) = lx - x^2$ on $(0, l)$.

i: Since f is a polynomial, $\int_0^l |f(x)|^2 dx < \infty$, so we have L^2 convergence.

ii: f, f' are continuous, so we have pointwise convergence.

iii. f, f', f'' are continuous and $f(0) = f(l) = 0$, so f satisfies the boundry conditions, so the series converges uniformly by Thm 2.

c: $f(x) = \frac{1}{x^2}$ on $(0, l)$. None of the conditions ~~are~~ of Thm's 2, 3, + 4 are met by f, so we cannot conclude any kind of convergence.

#12: We have $x = \sum_{n=1}^{\infty} \frac{2l(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{l}$.

The series converges to x in the L^2 sense by Thm 5.4.3, so by Thm 5.4.6 we have Parseval's equality:

$$\sum_{n=1}^{\infty} \left| \frac{2\ell(-1)^{n+1}}{n\pi} \right|^2 \int_0^\ell \left|\sin \frac{n\pi x}{\ell}\right|^2 dx = \int_0^\ell |x|^2 dx$$

Since $\int_0^\ell \sin^2 \frac{n\pi x}{\ell}\, dx = \frac{\ell}{2}$, the identity becomes

$$\sum_{n=1}^{\infty} \left(\frac{2\ell}{n\pi}\right)^2 \frac{\ell}{2} = \int_0^\ell x^2 dx = \frac{\ell^3}{3}$$

So $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

#16: Since $\|\phi(x)\| = \int_{-\pi}^{\pi} |x|^2 dx$

$= \frac{2}{\pi}\pi^2 < \infty$, the minimizing coefficients are just the Fourier coefficients by Theorem 5.4.5. Since ϕ is even the coefficients of sin are zero: $b_1 = b_2 = 0$.

We have $a_n = A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx$

$= \frac{2}{\pi}\int_0^\pi x \cos nx\, dx = \left(\frac{x}{n}\sin nx \Big|_0^\pi - \frac{1}{n}\int \sin nx\, dx\right)\frac{2}{\pi}$

$= +\frac{2}{n\pi}\left(\frac{\cos nx}{n}\Big|_0^\pi\right) = \frac{2}{n^2\pi}\left((-1)^n - 1\right)$, and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \, dx = \pi$$

So $f(x) = \frac{\pi}{2} - \frac{1}{\pi}\cos 2x$.