Section 6.1

#2: Using the Laplace operator in spherical coordinates, u must satisfy
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} u(r) \right) = \kappa^2 u(r) \Rightarrow r \frac{d}{dr} u''(r) + 2 u'(r) = \kappa^2 u(r) \cdot r. \]
Suppose u satisfies this equation and set \( v = u \cdot r \). The \[ \frac{d}{dr} v = u' \cdot r + u \cdot r, \] and \[ \frac{d^2}{dr^2} v = u'' \cdot r + 2 u' = \kappa^2 u \cdot r = \kappa^2 v \]
using the equation for u. Solving \( v'' - \kappa^2 v = 0 \) we find \( v = Ae^{\kappa r} + Be^{-\kappa r} \), so that \[ u = \frac{v}{r} = \frac{Ae^{\kappa r} + Be^{-\kappa r}}{r} \].

#3: Assume we look for functions \( u = u(r) \) that depend on \( r \) only. Using the polar coordinate form of \( \Delta \), we must solve \( u_{rr} + \frac{1}{r} u_r = \kappa^2 u \).
Bessel's differential equation is
\[ \frac{d^2}{dz^2} f(z) + \frac{1}{z} \frac{d}{dz} f(z) + \left( 1 - \frac{s^2}{z^2} \right) f(z) = 0, \]
where \( s \) is a constant. Its solutions are the Bessel functions. Consider the equation with \( s = 0 \):
\[ \frac{d^2}{dz^2} f(z) + \frac{1}{z} \frac{d}{dz} f(z) = -f(z). \]
Suppose $f(z)$ is a solution of $(\ast)$ and set $u(r) = f(ikr)$. Then
\[
\frac{d}{dr} u(r) = \frac{d}{dr} f(ikr) = f'(ikr) \cdot ik, \quad \text{and}
\]
\[
\frac{d^2}{dr^2} u(r) = f''(ikr)(ik)^2 = -k^2 f''(ikr). 
\]
So
\[
ur + \frac{i}{r} ur = -k^2 f''(ikr) + \frac{ik}{r} f'(ikr)
\]
by $(\ast) = -k^2 (-f(ikr)) = k^2 u(r).
\]
So $u(r) = f(ikr)$ is a solution to the original problem $\Delta u = k^2 u$. According to section 10.5, the functions $f$ that satisfy $(\ast)$ are just linear combinations of two Bessel functions $J_o$ and $J_o$. So we have solutions $u(r) = A J_o(ikr) + B J_o(ikr)$.

#5: Let's look for solutions $u$ that depend only on $r$; the PDE becomes an ODE:
\[
ur + \frac{1}{r} ur = 1 \iff r ur + ur = r
\]
i.e.
\[
\frac{d}{dr} (r ur) = r.
\]
Integrating, we have
\[
r ur = \frac{1}{2} r^2 + C_1 \implies ur = \frac{1}{2} r + \frac{C_1}{r}.
\]
\[
\implies u = \frac{1}{4} r^2 + C_1 \log r + C_2. \quad \text{Since $u$ must be defined for $r = 0$, we take $C_1 = 0$}. 
\]
The boundary condition gives
\[ O = U(a) = \frac{a^2}{4} + c_2, \quad \text{so} \]
\[ U = \frac{r^2}{4} - \frac{a^2}{4}. \]

#9 6a. Assuming the temperature \( T \) is a function of \( r \) only we have
\[ \Delta T = Tr'' + \frac{2}{r} Tr' = 0; \text{ this is an ODE,} \]
\[ \text{in fact we have } \frac{d}{dr} (r^2 Tr) = 2r Tr + r^2 Tr'. \]
\[ = r^2 (Tr' + \frac{2}{r} Tr) = 0. \]
\[ \text{Integrating gives } \]
\[ r^2 Tr = C_1 \Rightarrow Tr = C_1 r^{-2} \Rightarrow \]
\[ T = -C_1 r^{-2} + c_2. \] Now we apply the boundary conditions:
\[ 100 = T(1) = -C_1 + c_2 \Rightarrow C_2 = C_1 + 100, \text{ and} \]
\[ -8 = Tr(2) = C_1 2^{-2} = \frac{C_1}{4}. \]
\[ \text{So } C_1 = -48, \quad C_2 = -48 + 100, \text{ and} \]
\[ T(r) = 48 r^{-2} - 48 + 100. \]

b: The temperature decreases as \( r \) increases, so it's hottest at \( r = 1 \), \( u/ \)
\[ T(1) = 100, \text{ and coldest at } r = 2, \quad u/ \]
\[ T(2) = 100 - 48. \]

c: \( \gamma = 40 \)
We take the hint and guess that the solution has form \( u = Ax^2 + By^2 + Cxy + Dx + Ey + F \). Then \( \Delta u = 2A + 2B \), so Laplace's Eqn. implies that \( B = -A \). We apply the BC's:

1. \( U_x = 2Ax + Cy + D \)
2. \( U_y = 2By + Cx + E = -2Ay + Cx + E \).

- \( a = U_x(0, y) = Cy + D \Rightarrow C = 0 \) and \( D = -a \).
- \( a = U_x(a, y) = 2Aa + Cy + D = 2Aa - a = a(2A - 1) \)
  \( \Rightarrow A = \frac{1}{2} \)
- \( b = U_y(x, 0) = Cx + E \Rightarrow C = 0, \ E = b \)
- \( 0 = U_y(x, b) = 2Bb + Cx + E = 2Bb + b = b(2B + 1) \)
  \( \Rightarrow B = -\frac{1}{2}, \) which agrees with our computation \( B = -A \) above.

So \( u(x, y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax + by + F \), where \( F \) is arbitrary.
2. We show that \{ \sin m\pi x, \sin n\pi z \} are orthogonal on \( 0 \leq y \leq \pi, 0 < z < \pi \), by integrating the product of two such functions over the square:

\[
\int_0^\pi \int_0^\pi (\sin m\pi y, \sin n\pi z) (\sin m\pi y, \sin n\pi z) \, dy \, dz
\]

\[
= \left( \int_0^\pi \sin m\pi y \, dy \right) \left( \int_0^\pi \sin n\pi z \, dz \right)
\]

\[
= 0 \quad \text{unless} \quad m_1 = m_2 \quad \text{and} \quad n_1 = n_2 \quad \text{because the functions } \{ \sin kx \} \text{ are orthogonal on } 0 < x < \pi.
\]

4. There are two inhomogeneous boundary conditions; our strategy is to split the problem into two subproblems, each of which involves only one inhomogeneous boundary condition:

1. Solve \( \Delta u = 0 \) on \( D = \{ 0 < x < 1, 0 < y < 1 \} \)

\[
\begin{align*}
\text{BCs:} \quad & u(x, 0) = u(x, 1) = u_x(0, y) = 0, \quad u_x(1, y) = y^2 \\
\text{IVCs:} & \quad u_x = y^2, \quad u = 0
\end{align*}
\]

The PDE gives \( \frac{\partial^2 \xi}{\partial y^2} = -\lambda \xi \). We separate variables: \( u = \xi(x) \eta(y) \)

First we solve \( \eta'' = -\lambda \eta \), the BCs give \( 0 = \eta(0) = \eta(1) \), so \( \lambda = (n\pi)^2, \quad n > 0 \) with corresponding eigenfunctions \( \eta_n(x) = \sin n\pi x \).

Next we solve \( \xi'' = \lambda \xi \) with the single boundary condition \( \xi'(0) = 0 \). We have
\[
\sum_{n=1}^{\infty} \alpha_n \cosh n\pi x + B \sinh n\pi x, \quad \text{and}
\]
\[
O = \sum_{n=1}^{\infty} B_n \pi \cosh(\pi) = Bn\pi
\]
\[
\Rightarrow B = 0, \quad \text{so } \sum_{n=1}^{\infty} \alpha_n \cosh n\pi x.
\]
So \( u(x, y) = \sum_{n=1}^{\infty} \alpha_n \cosh n\pi x \). 

Sine satisfies the PDE and the three homogeneous BCs.

For the remaining side we require that
\[
y^2 = u_x(1, y) = \sum_{n=1}^{\infty} \left( A_n \cdot \pi \cdot \sinh n\pi x \right) \sin n\pi y
\]
Let \( y^2 = \sum_{n=1}^{\infty} C_n \sin n\pi y \) be the Fourier sine series for \( y^2 \). Choosing \( A_n = \frac{C_n}{\pi \cdot \sinh n\pi x} \), \( n \neq 0 \), \( \sum_{n=1}^{\infty} \frac{C_n}{\pi \cdot \sinh n\pi x} \)

2. Solve \( \Delta u = 0 \) on \( D \) with

\[
\begin{align*}
& u = 0 \\
& u_x = 0 \\
& u = x \\
\end{align*}
\]

\( u_x(0, y) = u_x(1, y) = u(x, 1) = 0, \quad u(x, 0) = x \).

Again, separate variables and obtain
\[
-\frac{\vec{X}''}{\vec{X}} = \frac{\vec{Y}''}{\vec{Y}} = \lambda.
\]

First we solve \( \vec{X}'' = -\lambda \vec{X} \); the BCs give \( \vec{X}'(0) = \vec{X}'(1) \). These are Neumann conditions, so \( \lambda = (n\pi)^2 \quad n \geq 0 \) with eigenfunctions \( \vec{X}_n = \text{constant} \), \( \vec{X}_n = \cos n\pi x \) for \( n \geq 0 \).

Now we solve \( \vec{Y}'' = \lambda \vec{Y} \) with the single boundary condition \( \vec{Y}(0) = 0 \).
For $\lambda = 0$ we have $\nabla'' = 0 \Rightarrow \nabla (y) = Ay + B$, and $O = \nabla (1) = A + B$, hence $B = -A$ and $\nabla (y) = A (y - 1)$. For $n > 0$, $\nabla'' = \lambda \nabla$ has solutions $\nabla = A \cosh \nu \pi x + B \sinh \nu \pi x$. Then $O = \nabla (1) = A \cosh \nu \pi x + B \sinh \nu \pi x$, so $B = -A \coth \nu \pi x$ and $\nabla (y) = A (\cosh \nu \pi x - \coth \nu \pi x \cdot \sinh \nu \pi x)$.

So $u(x, y) = \sum_{n=1}^{\infty} A_n [\cosh \nu \pi x - \coth \nu \pi x \cdot \sinh \nu \pi x] \cdot \cos \nu \pi x$, satisfies the PDE and the three homogeneous BCs. For the remaining BC, we require that

$$x = u(x, 0) = \sum_{n=1}^{\infty} A_n \cos \nu \pi x = -A_0 + \sum_{n=1}^{\infty} A_n \cos \nu \pi x.$$

Writing the Fourier cosine series for $x$:

$$x = \frac{1}{2} C_0 + \sum_{n=1}^{\infty} C_n \cos \nu \pi x,$$

we see that we must have $A_0 = -\frac{1}{2} C_0$ and $A_n = C_n$ for $n > 0$, where $C_n = \frac{2}{T} \int_0^T x \cos \nu \pi x \, dx$.

1. Let $u_1$ be the solution found in (1), and $u_2$ the solution found in (2). Let $u = u_1 + u_2$. Since $u_1$, $u_2$ harmonic, so is $u$. Moreover,
We get $\Sigma(y) = A \cosh n\pi y + B \sinh n\pi y$. The boundary condition gives $0 = \Sigma(0) = A \cosh n\pi (0) + B \sinh n\pi (0)$, so $B = -A \cotanh n\pi$, and

$\Sigma(y) = A (\cosh n\pi y - \cotanh n\pi \cdot \sinh n\pi y)$

for $n > 0$.

So $U(x, y) = \sum_{n=1}^{\infty} A_n \left[ \cosh n\pi x - \cotanh n\pi \cdot \sinh n\pi x \right] \cos n\pi x$

satisfies the PDE and the three homogeneous boundary conditions. For the remaining BC, we require

$x \cdot U(x, 0) = \sum_{n=1}^{\infty} A_n \cos n \pi x$

for $n = 0$
the sum satisfies the two homogeneous and two inhomogeneous BC's that we are looking for, so \( u \) is our solution.

\[ \frac{-\frac{\partial^2 u}{\partial x^2}}{\frac{\partial^2 u}{\partial y^2}} = \lambda. \]  
Separating variables, the PDE gives \( -\frac{\partial^2 u}{\partial x^2} = \lambda \frac{\partial^2 u}{\partial y^2} \). The BC's imply that \( \lambda = \frac{\pi^2}{\pi^2} \). The eigenvalue problem for \( \lambda \) has solutions \( \lambda = n^2 \) for \( n \geq 1 \) with eigenfunctions \( \phi_n(x) = \sin nx \). For \( \frac{\partial^2 u}{\partial y^2} = \lambda \frac{\partial^2 u}{\partial y^2} \), the solutions have form \( \psi_n(y) = Ae^{\lambda y} + Be^{-\lambda y} \). The condition at \( \infty \) implies that \( \psi_n \to 0 \) as \( \lambda y \to \infty \) so we must have \( A = 0 \). Then the function \( u(x,y) = \sum_{n=1}^{\infty} B_n e^{-\lambda y} \sin nx \) satisfies \( \Delta u = 0 \) and three of the four BC's. For the fourth BC we must have

\[ A (h(x)) = U(x,0) = \sum_{n=1}^{\infty} B_n \sin nx, \]  
so the \( B_n \) must be the Fourier sine coefficients for \( h(x) \): 

\[ B_n = \frac{2}{\pi} \int_0^{\pi} h(x) \sin nx \, dx. \]
b: If we omit the condition at \( \infty \), the problem is solved in the same way except we no longer throw out the term \( e^{ny} \) in the expression for \( Y_n \). Thus,

\[
U(x, y) = \sum_{n=1}^{\infty} (A_n e^{ny} + B_n e^{-ny}) \sin nx
\]

satisfies the PDE and the homogeneous BCs on the left and right sides. For the last BC, we require that

\[
h(x) = U(x, 0) = \sum_{n=1}^{\infty} (A_n + B_n) \sin nx,
\]

so that \( A_n + B_n \) must be the \( n^{th} \) Fourier sine coefficient for \( h \). The point here is that there are many ways to choose \( A_n + B_n \) so that this is the case, therefore the solution \( U \) is not unique. This does not violate the proof for uniqueness in Sec. 6.1 since that proof assumes that the region \( D \) is bounded.
HW #7 Solutions

Section 5.5  #s 2, 4, 12, 13, 14.

#2: We prove the Schwartz...
#1: a: By the Maximum Principle,
\[ \max_{\partial \Omega} u = \max_{\partial \Omega} (3 \sin 2\theta + 1) = 4. \]

b: By the mean value property,
\[ u(0,0) = \text{average value of } u \text{ on the circle of radius } 2 = \frac{1}{2\pi} \int_0^{2\pi} (3 \sin 2\theta + 1) \, d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} 3 \sin 2\theta \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} 1 \, d\theta \]
\[ = 0 + 1 \]

#2: According to Sec. 6.3, \( \Delta u = 0 \) on the disk has solution
\[ u = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta). \]
For the boundary condition, we require that
\[ 1 + 3 \sin \theta = u(a, \theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} a^n A_n \cos n\theta + a^n B_n \sin n\theta. \]
Comparing coefficients, we see that
\[ A_0 = 2, \quad a B_1 = \frac{3}{a}, \quad \text{and all other coefficients are zero}. \]

#4: We must show that
\[ P(r, \theta) = \frac{a^2 - r^2}{a^2 - 2ar \cos \theta + r^2} \]
is harmonic. In polar coordinates, the Laplace operator is
\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2 \sin^2 \Theta} \frac{\partial^2}{\partial \Theta^2} , \] so we must compute \( P_{rr} \), \( \frac{1}{r} P_r \), and \( \frac{1}{r^2} P_{\Theta \Theta} \) and show that they sum to zero. Good luck!