Random Walk Algorithms: Lecture 10
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In which the random walk evolution of a probability distribution inspires the solution to the heat equation with specified initial condition.

Initial conditions

The heat equation describes the time evolution of a function of position, so a well-defined problem is to compute the evolution of some initial condition, \( u(0, s) = f(s) \). To figure out how to do this using the heat kernel we found in Lecture 8, we draw inspiration again from the \( B = (X + 2I + X^{-1})/4 \) random walk on \( \mathbb{Z} \).

Suppose the initial probability distribution vector is not \( \hat{e}_0 \), but is rather \( \vec{u}_0 = \sum_r u_{0,r} \hat{e}_r \).

Then

\[
 u_{0,s} = \langle \vec{u}_0, \hat{e}_s \rangle = \langle \sum_r u_{0,r} \hat{e}_r, \hat{e}_s \rangle = \sum_r u_{0,r} \delta_{s,r},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the usual inner (dot) product and

\[
 \delta_{s,r} = \begin{cases} 
 1 & \text{if } s - r = 0; \\
 0 & \text{otherwise}
\end{cases}
\]

is the Kronecker delta. After one timestep,

\[
 u_{1,s} = \langle \vec{u}_1, \hat{e}_s \rangle = \langle B \sum_r u_{0,r} \hat{e}_r, \hat{e}_s \rangle = \sum_r u_{0,r} \langle B \hat{e}_r, \hat{e}_s \rangle = \sum_r u_{0,r} \frac{1}{4} (\delta_{s,r+1} + 2\delta_{s,r} + \delta_{s,r-1}),
\]

so, since \( B \) is symmetric, we can write

\[
 B\delta_{s,r} = \frac{1}{4} (\delta_{s,r+1} + 2\delta_{s,r} + \delta_{s,r-1}).
\]

Similarly, after \( t \) timesteps,

\[
 u_{t,s} = \langle \vec{u}_t, \hat{e}_s \rangle = \langle B^t \sum_r u_{0,r} \hat{e}_r, \hat{e}_s \rangle = \sum_r u_{0,r} \langle B^t \hat{e}_r, \hat{e}_s \rangle = \sum_r u_{0,r} \frac{1}{2^{2t}} \sum_{k=0}^{2t} \binom{2t}{k} \delta_{s,r+t-k},
\]

and we write

\[
 B^t \delta_{s,r} = \sum_{k=0}^{2t} \binom{2t}{k} \delta_{s,r+t-k}.
\]
These equations have continuous analogues. Suppose $f \in C_0^\infty(\mathbb{R})$. Then the analogue of (1) is:

$$u(0, s) = f(s) = \langle f, \delta_s \rangle = \int f(r) \delta(s - r)dr,$$

and the analogue of (2) is:

$$u(t, s) = \int f(r) \frac{1}{\sqrt{4\pi \alpha t}} e^{(s-r)^2/(4\alpha t)}dr.$$  

Combining functions in this way is often useful, and has a name:

**Definition.** Let $f$ and $g$ be integrable functions on $\mathbb{R}$. Then the *convolution* of $f$ with $g$ is

$$(f \ast g)(s) = \int f(r) g(s - r)dr.$$

**Theorem.** The convolution of $f$ with the Gaussian kernel given in (4) solves the initial value problem $u(0, x) = f(x)$ with $u(t, s)$ satisfying the heat equation.  

**Proof.** At $t = 0$, (4) becomes (3), so it satisfies the initial condition. For $t > 0$, the integrand in (4) is a continuous and differentiable function of $t$ so

$$\frac{\partial}{\partial t} \int f(r) \frac{1}{\sqrt{4\pi \alpha t}} e^{(s-r)^2/(4\alpha t)}dr = \int f(r) \frac{\partial}{\partial t} \left( \frac{1}{\sqrt{4\pi \alpha t}} e^{(s-r)^2/(4\alpha t)} \right)dr.$$  

Similarly,

$$\alpha \frac{\partial^2}{\partial s^2} \int f(r) \frac{1}{\sqrt{4\pi \alpha t}} e^{(s-r)^2/(4\alpha t)}dr = \int f(r) \alpha \frac{\partial^2}{\partial s^2} \left( \frac{1}{\sqrt{4\pi \alpha t}} e^{(s-r)^2/(4\alpha t)} \right)dr.$$  

But we already learned that the heat kernel satisfies the heat equation, so these two expressions are equal. 

\[\blacksquare\]