Random Walk Algorithms: Lecture 13
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In which the discrete Fourier transform is derived, and used to diagonalize the transition probability matrix for homogeneous random walks in one dimension.

The discrete Fourier transform

To diagonalize $X$, notice that $X^N = I$, so if $(\lambda, \vec{v})$ is an (eigenvalue, eigenvector) pair for $X$, i.e., $X \vec{v} = \lambda \vec{v}$, with $\vec{v} \neq 0$, then

$$\vec{v} = I \vec{v} = X^N \vec{v} = \lambda^N \vec{v},$$

so we can conclude $\lambda^N = 1$. That is, if we set $\omega = e^{2\pi i/N}$, the set of eigenvalues of $X$ is $\{\omega^k | k \in \{0, \ldots, N - 1\}\}$.

To find the corresponding eigenvectors we must solve:

$$0 = (X - \omega^k I) \vec{v} = \begin{pmatrix} -\omega^k & -\omega^k & \cdots & -\omega^k \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{pmatrix}.$$

Setting $v_0 = 1$, this implies $1 - \omega^k v_1 = 0$, so $v_1 = \omega^{-k}$. Then $\omega^{-k} - \omega^k v_2 = 0$, so $v_2 = \omega^{-2k}$, etc. Normalizing the eigenvectors to have norm 1 gives

$$\hat{f}_k = \frac{1}{\sqrt{N}} \begin{pmatrix} \omega^{-k} \\ \omega^{-2k} \\ \vdots \\ \omega^{-(N-1)k} \end{pmatrix}.$$  

Denoting conjugate transpose by $^\dagger$, we have

$$\hat{f}_j^\dagger \hat{f}_k = \frac{1}{N} \sum_{n=0}^{N-1} \omega^{nj} \omega^{-nk} = \frac{1}{N} \sum_{n=0}^{N-1} \omega^{n(j-k)} = \begin{cases} \frac{1}{N} & \text{if } j \neq k; \\ \frac{1}{1 - \omega^{-N} j^{-k}} & \text{if } j = k. \end{cases}$$

Since the $\hat{f}_k$ are the columns of the diagonalizing matrix,

$$F = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-(N-1)^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)} & \omega^{-(N-1)^2} & \cdots & \omega^{-(N-1)^2} \end{pmatrix}.$$
and $F^\dagger F = I = FF^\dagger$, so $F^{-1} = F^\dagger$, i.e., $F$ is unitary. $F$ is called the discrete Fourier transform, perhaps first written down in this form by Sylvester [1].

**Diagonalizing the transition probability matrix**

As we noted in Lecture 12, since $F$ diagonalizes $X$, it also diagonalizes $B$:

$$F^{-1}BF = \frac{1}{4}F^{-1}(X + 2I + X^{-1})F$$

$$= \frac{1}{4}\left(\begin{pmatrix}
1 & \omega^{-1} & \cdots & \omega^{-(N-1)} \\
\omega^{-1} & 2 & \cdots & 2 \\
\vdots & \ddots & \ddots & \vdots \\
\omega^{-(N-1)} & 2 & \cdots & 1
\end{pmatrix}\right) + \left(\begin{pmatrix}
1 & \omega^{-1} & \cdots & \omega^{-(N-1)} \\
\omega^{-1} & 2 & \cdots & 2 \\
\vdots & \ddots & \ddots & \vdots \\
\omega^{-(N-1)} & 2 & \cdots & 1
\end{pmatrix}\right),$$

so the set of eigenvalues of $B$ is

$$\left\{ \lambda_k = \frac{1}{4}(\omega^k + \omega^{-k} + 2) = \frac{1}{2}(\cos \frac{2\pi k}{N} + 1) \mid k \in \{0, \ldots, \lfloor N/2 \rfloor\} \right\}.$$  

Notice immediately that $\lambda_0 = 1$, with eigenspace spanned by $\hat{f}_0 = (1,1,\ldots,1)/\sqrt{N}$; $\lambda_k = \lambda_{N-k}$ for $0 < k < N/2$, with eigenspace spanned by $\hat{f}_k$ and $\hat{f}_{N-k}$; and $\lambda_{N/2} = 0$, if $N$ even, with eigenspace spanned by $\hat{f}_{N/2} = (1,-1,\ldots,1,-1)/\sqrt{N}$.

**References**