Random Walk Algorithms: Lecture 6
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In which random walks are illustrated and evolved, and a conserved quantity is discovered, leading to consideration of a new Markov process comprised of two steps of a random walk.

Diagrams

Here are the two random walks we have seen so far, on $\mathbb{Z}$ and on $\mathbb{Z}/\ell\mathbb{Z}$, with $\ell = 10$:

The vertices in these diagrams represent states in $S$ and the edges represent non-zero transition probabilities. To more completely capture the transitions we can label directed edges with the transition probabilities:
Time evolution

Since $P$ is a matrix, it acts on $\mathbb{R}^{|S|}$ by matrix multiplication. Suppose $\vec{u} \in \mathbb{R}^{|S|}$ with $u_x \geq 0$ for all $x \in S$ and $\sum_x u_x = 1$. Then $\vec{u}$ is a probability distribution on $S$. For $S = \mathbb{Z}/\ell\mathbb{Z}$, consider $\vec{u}_0 = (1, 0, \ldots, 0)$, indexed as the transition probability matrix is above; this is the probability distribution with $\Pr(S = 0) = 1$ and $\Pr(S \neq 0) = 0$. Then let

$$\vec{u}_1 = P\vec{u}_0 = \begin{pmatrix} 0 & 1 - p & p \\ p & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 - p & 0 \\ 1 - p & p & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ p \\ \vdots \\ 0 \end{pmatrix}.$$  

This is the probability distribution with $\Pr(S = 1) = p$; $\Pr(S = \ell - 1) = 1 - p$; and $\Pr(S \notin \{1, \ell - 1\}) = 0$, which is the probability distribution for the state after one step of the random walk. Similarly,

$$\vec{u}_2 = P\vec{u}_1 = P \cdot P\vec{u}_0 = P^2\vec{u}_0 = \begin{pmatrix} 2p(1 - p) \\ 0 \\ p^2 \\ 0 \\ \vdots \\ 0 \\ (1 - p)^2 \\ 0 \end{pmatrix},$$

which is the probability distribution over $S$ after two steps of the random walk. The general result is:

**Theorem.** Let $\vec{u}_0$ be the probability distribution for the initial state $X_0$ of a Markov process $\{X_t \mid t \in \mathbb{N}\}$. Then the probability distribution for $X_t$ is

$$\Pr(X_t = x) = (\vec{u}_t)_x = (P^t\vec{u}_0)_x.$$

A conserved quantity

Notice that if $\ell$ is even, the probability distributions at time 0 and time 2 above have nonzero probabilities only at even numbered states, while at time 1 the probability distribution has nonzero probabilities only at odd numbered states. This is true more generally:

**Lemma.** Let $X_0 = x_0$ for a random walk on $\mathbb{Z}$ or on $\mathbb{Z}/\ell\mathbb{Z}$, $\ell$ even. Then $x_t + t \pmod{2}$ is a “conserved quantity”, i.e., it is constant.

**Proof.** Consider the change in one timestep: $x_t + t \mapsto (x_t \pm 1) + (t + 1) \in \{x_t + t, x_t + t + 2\}$. But these two values are the same modulo 2, in $\mathbb{Z}$ or $\mathbb{Z}/\ell\mathbb{Z}$, for even $\ell$. 

2
This suggests considering a new Markov process with states $S = \{x \in 2\mathbb{Z}\}$, and transition matrix $P^2$, i.e., two steps of the random walk on $\mathbb{Z}$ we have been considering heretofore:

$$
P^2 = 
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-4 & (1-p)^2 & & & & & \\
-2 & 2p(1-p) & (1-p)^2 & & & & \\
0 & p^2 & 2p(1-p) & (1-p)^2 & & & \\
2 & p^2 & 2p(1-p) & & & & \\
4 & & & p^2 & \ddots & & \\
\vdots & & & & & \ddots & \\
\end{pmatrix}.
$$

We also will refer to such a Markov process, in which there are nonzero transition probabilities not only to adjacent states, but also to each state itself, as a random walk. Most generally, if there is some geometry on the space of states, and there are only nonzero transitions between nearby states, we will call the Markov process a random walk. In the case $p = 1/2$, the transition probability matrix above becomes

$$
B = 
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-2 & 1/4 & & & & & \\
0 & 1/2 & 1/4 & & & & \\
2 & 1/4 & & & & & \\
\vdots & & & & & & \ddots \\
\end{pmatrix} = 
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-2 & \ddots & 1 & & & & \\
1/4 & 0 & 2 & & & & \\
2 & 1 & \ddots & & & & \\
\vdots & & & & & & \ddots \\
\end{pmatrix},
$$

which we call $B$ for “binomial”. Setting $\vec{u}_0 \in \mathbb{R}^{2\mathbb{Z}}$ to be 1 only in the 0th component, the probability distribution after $t$ steps of this new random walk is

$$
\vec{u}_t = B^t \vec{u}_0.
$$

We will refer to the original random walk as $\text{RW}^1(p)$, and to the random walk with transition probability matrix $P^2$ as $\text{RW}^2(p)$. 

3