Random Walk Algorithms: Lecture 7
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In which the heat equation with drift in one dimension is derived as a continuum limit of a random walk on $2\mathbb{Z}$, and the advection equation is solved.

Difference equations

Having constructed a new random walk from two steps of the original random walk in the last lecture, we have an equation for the evolution of the probability distribution over $2\mathbb{Z}$ during one timestep:

$$\tilde{u}_{t+1} = B\tilde{u}_t.$$  

Written as an equation for the change in the probability distribution, this becomes:

$$\begin{bmatrix} \ldots & -2 & 0 & 2 & \ldots \\ \vdots \\ -2 & \ddots & \ddots & \ddots & -2 \\ 0 & 1 & 2 & 1 & \ddots \\ 2 & 1 & \ddots & \ddots & \ddots \\ \vdots \\ \ldots & 2 & 0 & -2 & \ldots \end{bmatrix} \begin{bmatrix} \tilde{u}_t \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \ldots & -2 & 0 & 2 & \ldots \\ \vdots \\ -2 & \ddots & \ddots & \ddots & -2 \\ 0 & 1 & 2 & 1 & \ddots \\ 2 & 1 & \ddots & \ddots & \ddots \\ \vdots \\ \ldots & 2 & 0 & -2 & \ldots \end{bmatrix} \begin{bmatrix} \tilde{u}_t \end{bmatrix} = \frac{1}{4} L \tilde{u}_t,$$

where $L$ is called the graph Laplacian and is defined as the adjacency matrix for a graph (the matrix $A$ indexed by the vertices of a graph with $A_{ij} = 1$ if there is an edge connected $i$ and $j$ and $A_{ij} = 0$ otherwise) minus the diagonal matrix $D$ with $D_{ii}$ being the degree of $i$: $L = A - D$. Written in terms of components, this becomes a set of coupled difference equations:

$$u_{t+1,x} - u_{t,x} = \frac{1}{4}(u_{t,x+2} - 2u_{t,x} + u_{t,x-2}).$$

For each $t$, $\{u_{t,x} \mid x \in 2\mathbb{Z}\}$ is a probability distribution. We can imagine that it is a discrete set of values of a differentiable function. The two blue line segments shown not only approximate the function on the intervals $[x - \Delta x, x]$ and $[x, x + \Delta x]$, but their slopes also approximate the left and right derivatives at $x$. 
Derivatives

Recall that the right and left derivatives of a function \( f : \mathbb{R} \to \mathbb{R} \) are defined as the limits:

\[
\begin{align*}
    f'_+ (x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}; \\
    f'_- (x) &= \lim_{\Delta x \to 0} \frac{f(x) - f(x - \Delta x)}{\Delta x}.
\end{align*}
\]

If these two limits are equal, then the function is differentiable at \( x \). Supposing \( f \) is also second differentiable at \( x \), its second derivative is

\[
\begin{align*}
    f''(x) &= \lim_{\Delta x \to 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x} \\
    &= \lim_{\Delta x \to 0} \frac{(f(x + \Delta x) - f(x))/\Delta x - (f(x) - f(x - \Delta x))/\Delta x}{\Delta x} \\
    &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{(\Delta x)^2}.
\end{align*}
\]

If the duration of each timestep is \( \Delta t \), and the length of each spatial unit is \( \Delta x \), we can rewrite the difference equation above as:

\[
\begin{align*}
    u_{t+\Delta t,x} - u_{t,x} &= \frac{1}{4}(u_{t,x+\Delta x} - 2u_{t,x} + u_{t,x-\Delta x}),
\end{align*}
\]

whence the definitions of derivatives suggest setting \( \Delta t = (\Delta x)^2/(4\alpha) \), \( 0 < \alpha \in \mathbb{R} \), dividing, and taking the limit as \( \Delta t \to 0 \):

\[
\lim_{\Delta t \to 0} \frac{u_{t+\Delta t,x} - u_{t,x}}{\Delta t} = \lim_{\Delta x \to 0} \frac{1}{4} \frac{u_{t,x+\Delta x} - 2u_{t,x} + u_{t,x-\Delta x}}{(\Delta x)^2/(4\alpha)}
\Rightarrow \quad \frac{\partial u(t,x)}{\partial t} = \alpha \frac{\partial^2 u(t,x)}{\partial x^2}.
\]

This is the diffusion (or heat) equation.

Drift

It is only for \( p = 1/2 \) that two steps of the initial random walk are described by the transition matrix \( B \); otherwise, let \( p = 1/2 + \epsilon \), for \(-1/2 \leq \epsilon \leq 1/2\). Then the elements in the two step transition probability matrix \( P^2 \) are:

\[
\begin{align*}
    p^2 &= \left(\frac{1}{2} + \epsilon\right)^2 = \frac{1}{4} + \epsilon^2 + 2\epsilon \\
    2p(1-p) &= 2\left(\frac{1}{2} + \epsilon\right)\left(\frac{1}{2} - \epsilon\right) = \frac{1}{2} - 2\epsilon^2 \\
    (1-p)^2 &= \left(\frac{1}{2} - \epsilon\right)^2 = \frac{1}{4} + \epsilon^2 - 2\epsilon.
\end{align*}
\]
so the difference equations become

\[ u_{t+1,x} - u_{t,x} = \left(\frac{1}{4} + \epsilon^2\right)(u_{t,x+2} - 2u_{t,x} + u_{t,x-2}) - 2\epsilon(u_{t,x+2} - u_{t,x-2}). \]

As before we want to divide the left hand side by \( \Delta t \), and the first term on the right hand side by \( (\frac{1}{4} + \epsilon^2)(\Delta x)^2/\alpha \). For the last term to become a derivative in the limit we must divide it by \( 2\Delta x \) (since it contains \( u_{t,x+\Delta x} - u_{t,x-\Delta x} \), and to clean up the coefficient in front of the derivative we will divide by \( 2\epsilon \cdot 2\Delta x/\beta \). Thus we must set

\[ \Delta t = \left(\frac{1}{4} + \epsilon^2\right)\frac{1}{\alpha}(\Delta x)^2 = \frac{4\epsilon}{\beta} \Delta x, \]

(which means \( \beta \) has the same sign as \( \epsilon \)) divide by this quantity, and take the limit as it goes to 0. (Notice that this means that \( \epsilon \to 0 \).) The result is the heat equation with \textit{drift}:

\[ \frac{\partial u(t, x)}{\partial t} = \alpha \frac{\partial^2 u(t, x)}{\partial x^2} - \beta \frac{\partial u(t, x)}{\partial x}. \]

\textbf{The advection equation}

Setting \( \alpha = 0 \) in the heat equation with drift gives the \textit{advection} equation:

\[ \frac{\partial u(t, x)}{\partial t} = -\beta \frac{\partial u(t, x)}{\partial x}. \]

If we define new variables \( (\tau, \chi) = (t, x - \beta t) \), then

\[ \frac{\partial u}{\partial \tau} = \frac{\partial u}{\partial t} \frac{\partial t}{\partial \tau} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial \tau} = \frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = 0, \]

using the chain rule, and then the advection equation. Thus

\[ u(t, x) = u(t(0, \chi), x(0, \chi)) = u(0, \chi) = u(0, x - \beta t), \]

so whatever the initial function \( u(0, x) \) is, at time \( t \) it is merely shifted by \( \beta t \).