168A MIDTERM

In problem 1, write pseudocode for your algorithms; you can also describe in words what they are doing. In problems 2 and 3, showing your work in addition to the final answer might earn some partial credit in case the latter is not exactly right.

1. a. [15 points] Construct an algorithm to select a point uniformly at random from the disk of radius 1 centered at the origin. Assume you have access to a random number generator random() that returns a uniformly random number in the interval \([0, 1]\), e.g., \(x \leftarrow \text{random()}\) means that \(x \in [0, 1]\), sampled from the constant probability density function.

b. [20 points] Now construct an algorithm to approximate the area of a disk of radius 1, i.e., to approximate \(\pi\). Design your algorithm so that it has probability at least \(2/3\) of giving an estimate for \(\pi\) that is correct to one decimal place, i.e., it is within the interval \([\pi - 0.05, \pi + 0.05]\).

Hint: For large \(n\) and fixed \(p\), you can approximate a binomial distribution by a normal distribution with the same mean and variance, i.e., \(\text{Binomial}(n, p) \approx \mathcal{N}(np, np(1 - p))\).

2. Consider a random walk \(\{X_t \in \mathbb{Z} \mid t \in \mathbb{N}\}\) with transition probabilities

\[P_{xy} = \begin{cases} 1/2 & \text{if } |x - y| = 1; \\ 0 & \text{otherwise}, \end{cases}\]

and initial state \(X_0 = 0\).

a. [15 points] Suppose \(X_4 = 2\). Draw all the paths in \((x, t)\) space that the walk could follow.

Hint: Each path is like the path that you plotted in the first homework assignment.

b. [20 points] Compute the probability distributions for \(X_1, X_2,\) and \(X_3\) given that \(X_0 = 0\) and \(X_4 = 2\), i.e., compute \(\Pr(X_t = x \mid X_0 = 0, X_4 = 2)\) for \(t \in \{1, 2, 3\}\).

3. a. [15 points] Write the transition probability matrix \(P\) for a random walk on \(S = \{1, 2, 3, 4, 5, 6\} \subseteq \mathbb{N}\) with transition probabilities:

\[
\Pr(X_{t+1} = x \mid X_t = y) = \begin{cases} 1/2 & \text{if } x = y; \\ 1/2 & \text{if } y \in \{1, 6\} \text{ and } |x - y| = 1; \\ 1/4 & \text{if } y \not\in \{1, 6\} \text{ and } |x - y| = 1; \\ 0 & \text{otherwise.} \end{cases}
\]

b. [15 points] Find a probability distribution on \(S\) that is unchanged by the evolution.