7 January 2008

MATH 180A. INTRODUCTION TO PROBABILITY LECTURE 1

David A. Meyer

Project in Geometry and Physics, Department of Mathematics
University of California/San Diego, La Jolla, CA 92093-0112
http://math.ucsd.edu/~dmeyer/; dmeyer@math.ucsd.edu

Games

Example 1. The dealer will pay \$1 if you flip a coin and it lands head up. How much will you pay to play this game? Most people in the class agreed that they would pay up to \$0.50.

Example 2. The dealer will pay \$2 if you roll a die and it lands with a 6 up. How much will you pay to play this game? Most people in the class agreed that they would pay up to \$0.33.

Example 3. The dealer will pay \$2 if the card you draw has a rank at least as high as the rank of the card he draws. How much will you pay to play this game? One person in the class was willing to pay \$1.07.

Defining 'probability'

Not only do these simple games recall the origins of the mathematical analysis of probability in attempts to optimize play in gambling games, but also the wagers people are willing to make demonstrate Bruno de Finetti's *definition* of probability [1]:

> *Supposons qu'un individu soit obligé d'évaluer le prix* p *pour lequel il serait disposé d'échanger la possession d'une somme quelconque* S *(positive ou négative) subordonée à l'arrivée d'un événement donné,* E*, avec la possession de la somme* pS*; nous dirons par définition que ce nombre* p *est la mesure du degré de probabilité attribué par l'individu considéré à l'événement* E*, ou, plus simplement, que* p *est la probabilité de* E *(selon l'individu considéré; cette précision pourra d'ailleurs être sous-entendue s'il n'y a pas d'ambiguïté).*

> Let us suppose that an individual is obliged to evaluate the rate p at which he would be willing to exchange the possession of an arbitrary sum S (positive or

> negative) contingent on the occurrence of a given event, E, for the possession of the sum pS; we will say by definition that this number p is the measure of the degree of probability attributed by this individual to the event E, or, more simply, that p is the probability of E (according to this individual; this specification can be implicit if there is no ambiguity).

That is, if someone is willing to pay up to pS dollars for the opportunity to win S dollars if something happens, then the *probability* of that event is p, according to that person. We will discuss this definition further in the next lecture.

From symmetry to probability

But first, let us return to the three games in the examples. According to this definition, in Example 1 the people in the class assigned a probability of 0.5 to the event that a coin lands head up when it is flipped. Can we understand why we believe this?

In principle, according to classical physics, we should be able to *predict* how the coin will land: if we know its initial position, the force exerted upon it by the flipper, the position of the surface on which it lands, the material properties of the coin and that surface, the air pressure, any winds that blow through the trajectory of the coin, *etc.*, we can apply Newton's laws of motion and the law of gravity, while accounting for the elasticity of the collisions between the coin and the surface, and for air friction, and for whatever other physical effects there are, to calculate the coin's motion until it stops. In practice, of course, these calculations are too difficult for us to do exactly, at least in our heads, even if we knew all the relevant factors. And furthermore, we do not know all of these factors.

But there is a *symmetry* in the problem—none of these factors interacts in a way that differs significantly with the different sides of the coin. Thus the amount we are willing to pay to win \$1 if the coin lands head up is the same amount that we are willing to pay to win \$1 if the coin lands tail up. Since we believe that coin will land either head up or tail up (which implies we would be willing to pay up to \$1 to win \$1 as long as the coin lands either head up or tail up), we calculate $p + p = 1$, so $p = 0.5$.

An analogous argument tells us that for a rolled die, the amount we would pay to win \$2 if the die lands with k up, for $k \in \{1, 2, 3, 4, 5, 6\}$, does not depend on the value of k. Thus $p+p+p+p+p+p = 1$, so $p = 1/6$. Rounding down to the nearest cent, $\$2 \cdot 1/6$ is \$0.33, which is how much most people in the class were willing to pay.

In Example 3, a similar argument tells us that as long as the cards are well shuffled, the symmetry of the problem leads us to assign equal probabilities to each of the possible outcomes, *i.e.*, the pairs of cards drawn in the game. Since there are 52 possible cards the dealer can draw, and then 51 possible cards for the player to draw, there are $52 \cdot 51$ possible outcomes so the probability of each is $1/(52 \cdot 51)$. Once the dealer draws a card, there are 3 more cards in the deck with the same value, so there are $52 \cdot 3$ outcomes in which the two cards have the same value (and hence the player wins). Of the remaining $52 \cdot 51 - 51 \cdot 3 = 52 \cdot 48$ outcomes, the player wins half, $52 \cdot 24$. So the total number of outcomes in which the player wins is $52 \cdot 3 + 52 \cdot 24 = 52 \cdot 27$. Adding the probabilities of each gives $52 \cdot 27/(52 \cdot 51) = 27/51$. Since $\$2 \cdot 27/52 = 1.03846$, the \$1.07 one student was willing to pay gives only a slight overestimate of this probability.

Vocabulary of gambling

DEFINITION. The amount, S, that the player can win in one of these games is the *total stake*. The amount, pS, the player pays to play the game is the *player's stake*. The difference, $S - pS = (1-p)S$, is the amount that the dealer is putting up; this is the *dealer's stake*. The player's *odds* of winning is the ratio of the player's stake to the dealer's stake: $p/(1-p)$. This is often phrased as "$p/(1-p)$ to 1". If $p = m/n$, where m and n are relatively prime (natural numbers with no common divisor greater than 1), the odds are often described as "m to $n - m$". These are sometimes written as "$p/(1-p) : 1$" and "$m : n - m$", respectively.

In Example 1 the odds for the player are $1 : 1$. In Example 2 they are $1 : 5$. In Example 3 they are $27 : 25$.

Notice that if the odds for winning are $a : b$ (or a/b), then the odds *against* winning are $b : a$ (or b/a).

References

[1] B. de Finetti, "*La prévision: ses lois logiques, ses sources subjectives*", *Annales de l'Institute Henri Poincaré* **7** no. 1 (1937) p. 6.