

# MATH 180A. INTRODUCTION TO PROBABILITY

## LECTURE 4

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### Probability distributions

Given an outcome space  $\Omega$ , a probability distribution is a function  $P : \mathcal{P}(\Omega) \rightarrow \mathbb{R}$ , where  $P(E)$  is the probability assigned to an event  $E \subseteq \Omega$ . In Lecture 1 [1] we learned de Finetti's definition of probability [2]:

DEFINITION. Suppose that for any stake  $S \in \mathbb{R}$ , an individual will pay  $pS$  to win  $S$  if  $E$  occurs. Then  $P(E) = p$ .

There are relations among events that we should expect to constrain the possible probability distributions. For example, if  $A \subseteq B \subseteq \Omega$ , we might expect  $P(A) \leq P(B)$ . These constraints follow from an assumption that individuals are rational [2]:

ASSUMPTION. An individual assigns probabilities to events *coherently*, *i.e.*, in such a way that the individual is never sure to lose.

This seemingly obvious assumption has surprisingly specific consequences.

PROPOSITION 1. *If  $P$  is a coherent probability distribution,  $P(\Omega) = 1$ .*

*Proof.* Suppose a stake  $S$  is paid if  $\Omega$  occurs, and an individual will pay  $pS$  to play. Since  $\Omega$  must occur—by definition it is the set of all possible outcomes—the individual “wins”

$$w = S - pS = (1 - p)S. \quad (1)$$

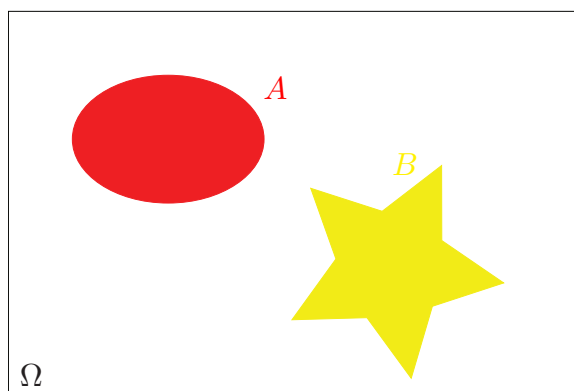
For any  $w \in \mathbb{R}$ , including  $w < 0$ , we can solve equation (1) for  $S$ , unless  $1 - p = 0$ . That is, unless  $p = 1$ , there is some  $S$  such that the individual is sure to lose money. Under the assumption that the individual assigns probabilities coherently, this means that  $P(\Omega) = p = 1$ . ■

Notice that if  $p < 1$  and  $w < 0$ , then the solution to equation (1) is some  $S < 0$ . We should think about the meaning of the wager in de Finetti's definition of probability when  $S < 0$ . In this case the individual is "paying"  $-p|S|$  to "win"  $-|S|$ . "Paying" a negative amount means receiving a positive amount from the other party in the wager, while "winning" a negative amount means paying out the absolute value of that amount. Thus when  $S < 0$  the parties to the wager change places: the first individual accepts a bet of  $p|S|$  from the other party, and pays out  $|S|$  if the event occurs. So the condition that the stake  $S$  in de Finetti's definition can be positive or negative means that the individual is willing to take either side of the wager with odds  $p : 1 - p$  for  $E$  occurring.

**PROPOSITION 2.** *If  $P$  is a coherent probability distribution,  $A, B \subseteq \Omega$ , and  $A \cap B = \emptyset$ , then  $P(A \cup B) = P(A) + P(B)$ .*

*Proof.* Let  $p_A = P(A)$ ,  $p_B = P(B)$ , and  $p = P(A \cup B)$ . Suppose stakes  $S_A$ ,  $S_B$  and  $S$  are wagered on the occurrence of  $A$ ,  $B$ , and  $A \cup B$ , respectively.

As shown in the figure,  $A\bar{B} = A \setminus B$ ,  $\bar{A}B = B \setminus A$ , and  $\bar{A}\bar{B} = \overline{A \cup B}$  form a partition of  $\Omega$ : these events have pairwise empty intersections and their union is  $\Omega$ . Thus exactly one of these three events occurs, and we can calculate the individual's "winnings" in each case:



$$\begin{aligned} A\bar{B} : \quad w_{A\bar{B}} &= (1 - p_A)S_A - p_B S_B + (1 - p)S \\ \bar{A}B : \quad w_{\bar{A}B} &= (1 - p_A)S_A - p_B S_B + (1 - p)S \\ \bar{A}\bar{B} : \quad w_{\bar{A}\bar{B}} &= -p_A S_A - p_B S_B - pS \end{aligned} \tag{2}$$

For any  $w_{A\bar{B}}$ ,  $w_{\bar{A}B}$ , and  $w_{\bar{A}\bar{B}}$ , including negative values, we can solve the system of linear equations (2) for  $S_A$ ,  $S_B$ , and  $S$ , unless the system is singular, *i.e.*, unless

$$0 = \begin{vmatrix} 1 - p_A & -p_B & 1 - p \\ -p_A & 1 - p_B & 1 - p \\ -p_A & -p_B & -p \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -p_A & -p_B & -p \end{vmatrix} = -p + p_B + p_A.$$

That is, unless  $p = p_A + p_B$ , there are some stakes  $S_A$ ,  $S_B$ , and  $S$  such that the individual is sure to lose money. Under the assumption that the individual assigns probabilities coherently, this means that  $P(A \cup B) = p = p_A + p_B = P(A) + P(B)$ . ■

The conclusions of Propositions 1 and 2,  $P(\Omega) = 1$  and  $P(A \cup B) = P(A) + P(B)$  if  $A \cap B = \emptyset$ , are the conditions required for  $P$  to be a probability distribution in Kolmogorov's formulation of probability [3]. We have derived them from de Finetti's assumption of coherence.

## References

- [1] <http://math.ucsd.edu/~dmeyer/teaching/180Awinter08/notes01.pdf>.
- [2] B. de Finetti, “*La prévision: ses lois logiques, ses sources subjectives*”, *Annales de l’Institut Henri Poincaré* **7** no. 1 (1937).
- [3] A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung* (1933); translation edited by N. Morrison, *Foundations of the Theory of Probability* (New York: Chelsea 1956).