5.7

Let \( k \in \mathbb{R} \), then

\[
F_Y(k) = \mathbb{P}(Y \leq k) = \mathbb{P}(X \leq e^k) = \int_0^{e^k} \lambda e^{-\lambda t} dt = 1 - e^{-\lambda e^k}.
\]

Thus, for all \( k \in \mathbb{R} \),

\[
f_Y(k) = F'_Y(k) = \lambda e^{k - \lambda e^k}.
\]

5.14

Since \( X \) takes value from \( \{0, 1, 2, 3, 4\} \), \( Y = (X - 2)^2 \) takes value from \( \{0, 1, 4\} \). Then,

\[
\mathbb{P}(Y = 0) = \mathbb{P}(X = 2) = \binom{4}{2} \left(\frac{1}{2}\right)^4 = \frac{3}{8},
\]

\[
\mathbb{P}(Y = 1) = \mathbb{P}(X = 1) + \mathbb{P}(X = 3) = \binom{4}{1} \left(\frac{1}{2}\right)^4 + \binom{4}{3} \left(\frac{1}{2}\right)^4 = \frac{1}{2},
\]

and

\[
\mathbb{P}(Y = 4) = \mathbb{P}(X = 0) + \mathbb{P}(X = 4) = \binom{4}{0} \left(\frac{1}{2}\right)^4 + \binom{4}{4} \left(\frac{1}{2}\right)^4 = \frac{1}{8}.
\]

Thus, the probability mass function for \( Y \) is

\[
p_Y(k) = \begin{cases} 
\frac{3}{8} & \text{if } k = 0 \\
\frac{1}{2} & \text{if } k = 1 \\
\frac{1}{8} & \text{if } k = 4 \\
0 & \text{otherwise} 
\end{cases}
\]

5.21

For \( t \in \mathbb{R} \),

\[
M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(aX+b)}] = e^{tb} \mathbb{E}[e^{(ta)X}] = e^{tb} M_X(at).
\]
6.3

Let $X_1$ be the number of times that she choose white chalk in the next 10 days, $X_2$ be the number of times that she choose yellow chalk in the next 10 days, and $X_3$ be the number of times that she choose purple chalk in the next 10 days. Then, $(X_1, X_2, X_3) \sim \text{Mult}(10, 3, 0.5, 0.4, 0.1)$.

(a) 
\[ P((X_1, X_2, X_3) = (5, 4, 1)) = \frac{10!}{5!4!1!} 0.5^5 0.4^4 0.1^1 = \frac{63}{625}. \]

(b) Since $X_1 \sim \text{Bin}(10, 0.5)$,
\[ P(X_1 = 9) = \binom{10}{9} 0.5^{10} = \frac{5}{512}. \]

6.7

(a) Let $k \in (0, 1)$. Then,
\[ F_Y(k) = P(Y \leq k) = \frac{\text{area of the trapezoid below the line } y=k}{\text{area of the whole triangle}} = 2k - k^2. \]

Thus, the marginal probability density for $Y$ is
\[ f_Y(k) = \begin{cases} 2 - 2k & \text{if } k \in (0, 1) \\ 0 & \text{o.w.} \end{cases} \]

By the symmetry of $X$ and $Y$ in this problem, the marginal probability density for $X$ is the same function.

(b) 
\[ \mathbb{E}[X] = \int_0^1 k(2 - 2k) \, dk = \frac{1}{3}, \]

and 
\[ \mathbb{E}[X] = \mathbb{E}[Y] = \frac{1}{3}. \]

(c) Since the joint density $f$ is constant on the triangle and integrate to 1,
\[ f(x, y) = \begin{cases} 2 & \text{if } (x, y) \text{ in the triangle} \\ 0 & \text{otherwise}. \end{cases} \]
Thus,

\[
\mathbb{E}[XY] = \int \int_{\Delta} 2xydA \\
= \int_{0}^{1} \int_{0}^{1-x} 2xydydx \\
= \frac{1}{12}.
\]

6.22

The random variable \(X_1 + X_2\) counts the number of times that outcome 1 or 2 appears among the \(n\) trails. Considering outcome 1 or 2 as a success and any other outcome as a failure, we see that \(X_1 + X_2\) follows Bin\((n, p_1 + p_2)\).

5.40

Suppose for contradiction that \(X\) is a random variable with \(M_X(1) = 3\) and \(M_X(2) = 4\). Let \(Y = e^X\). Then, \(\mathbb{E}[Y] = \mathbb{E}[e^X] = M_X(1) = 3\), and \(\mathbb{E}[Y^2] = \mathbb{E}[e^{2X}] = M_X(2) = 4\). Then, \(\text{var}(Y) = 4 - 3^2 = -5 < 0\), contradiction.