Please simplify your answers to the extent reasonable without a calculator. Show your work. Explain your answers, concisely.

1. Two bugs start at opposite corners of a square. During each minute they each walk along an edge, chosen uniformly at random, to an adjacent corner. If they run into each other on an edge they stop, and if they end at the same corner, they stop. Let $T$ be the time at which they stop.

   a. [15 points] Describe this situation as a discrete time Markov chain: specify a set of states for the system and write down the one step probability transition matrix.

   Let $X_t = \text{how far apart the bugs are at time } t$, counted by numbers of edges one must traverse to reach the other $\in \{0, 1, 2\}$. Then

   $$P = \begin{pmatrix}
   0 & 1 & 2 \\
   0 & 1 & 0 & 0 \\
   2 & 1/2 & 0 & 1/2 \\
   \end{pmatrix}.$$ 

   b. [10 points] What is $E[T]$?

   Let $\nu_i = E[T|X_0 = i]$. Then we have

   $$\nu_0 = 0$$
   $$\nu_1 = 1 + \nu_1 \cdot \frac{3}{4} \Rightarrow \nu_1 = 4$$
   $$\nu_2 = 1 + \nu_2 \cdot \frac{1}{2} \Rightarrow \nu_2 = 2$$

   so since the bugs start at opposite corners, $E[T] = 2$.

   c. [10 points] What would $E[T]$ be if the bugs start on adjacent vertices?

   As calculated above, in this case, $E[T] = \nu_1 = 4$.

2. [30 points] Suppose that in California voters register each year as members of the Republican, Democratic, or Green parties, and that they change their registration randomly each year with transition probability matrix

   $$
   \begin{pmatrix}
   R & D & G \\
   R & (0.85 & 0.15 & 0) \\
   D & (0.05 & 0.85 & 0.10) \\
   G & (0 & 0.05 & 0.95) \\
   \end{pmatrix}.
   $$

   In the long run, what fraction of voters will belong to each party?

   This transition matrix is regular so the limiting distribution is the stationary distribution, which satisfies:

   $$\pi_R = 0.85\pi_R + 0.05\pi_D \Rightarrow \pi_R = \pi_D/3$$
   $$\pi_G = 0.10\pi_D + 0.95\pi_G \Rightarrow \pi_G = 2\pi_D$$
Since $1 = \pi_R + \pi_D + \pi_G = (1/3 + 1 + 2)\pi_D$, we find $\pi = (1/10, 3/10, 6/10)$.

3. Suppose when people get the flu and don’t stay home in bed they infect 2 other people with probability $p > 1/2$ and infect no one with probability $1 - p$.

a. [15 points] What is the probability, starting with a single person with the flu, that the flu epidemic will die out?

Setting the extinction probability $u$ equal to the probability generating function gives

$$u = 1 - p + pu^2 \Rightarrow 0 = pu^2 - u + 1 - p = (u - 1)(pu - (1 - p))$$

$$\Rightarrow u = \frac{1 - p}{p} < 1 \quad \text{for } p > \frac{1}{2}.$$ 

b. [20 points] If we could change people’s behavior so that when they get sick, they stay home in bed with probability $q > 0$. How big must $q$ be (in terms of $p$) to make the extinction probability for the flu be 1?

In this case the extinction probability satisfies

$$u = q + (1 - q)(1 - p) + (1 - q)pu^2 \Rightarrow 0 = (1 - q)pu^2 - u + q + (1 - q)(1 - p)$$

$$= (u - 1)((1 - q)pu - [q + (1 - q)(1 - p)])$$

$$\Rightarrow u = \frac{q + (1 - q)(1 - p)}{(1 - q)p}.$$ 

Setting $u = 1$ gives $q = \frac{2p - 1}{2p}$.

4. [Extra credit: 25 points] Consider the Markov chain describing aging with reincarnation: it has states $X_n \in \mathbb{N}$; at state $i \in \mathbb{N}$ there is probability $p_i$ of living one more year, to state $i + 1$, and probability $1 - p_i$ of dying and returning to state 0. Describe conditions on the $p_i$ that ensure that the state 0 is recurrent.

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 & \cdots \\ 0 & 1 - p_0 & p_0 & & \\ 1 & 1 - p_1 & p_1 & & \\ 2 & 1 - p_2 & p_2 & & \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$ 

0 is recurrent if

$$1 = f_{00} = \sum_{n=0}^{\infty} f_{00}^{(n)}$$

$$= 1 - p_0 + p_0(1 - p_1) + p_0p_1(1 - p_2) + \cdots$$

which is true if there is some $n$ such that $p_n = 0$ or if for all $N > 0$ there is some $n > N$ such that $p_n < 1$. Otherwise there would be some $N$ such that for all $n > N$, $p_n = 1$, which would mean that reaching $N + 1$ would assure immortality, and this would happen with some positive probability.