Problem 1 (Pinsky & Karlin, Exercise 4.1.3). A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$P = \begin{pmatrix}
0 & 0 & 1 \\
0.1 & 0.1 & 0.8 \\
0.2 & 0.2 & 0.6 \\
0.3 & 0.3 & 0.4 \\
\end{pmatrix}$$

What fraction of time, in the long run, does the process spend in state 1?

Solution. Notice that $P$ is a regular transition probability matrix, so a limiting distribution $\pi = (\pi_0, \pi_1, \pi_2)$ exists and is the unique solution to the equations

$$\begin{align*}
\pi_0 &= 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2, \\
\pi_1 &= 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2, \\
\pi_2 &= 0.8\pi_0 + 0.6\pi_1 + 0.4\pi_2, \\
\pi_0 + \pi_1 + \pi_2 &= 1.
\end{align*}$$

Solving this system yields $\pi = (3/13, 3/13, 7/13)$. Thus, the long run mean fraction of time that the process spends in state 1 is $\pi_1 = 3/13$.  

Problem 2 (Pinsky & Karlin, Exercise 4.1.8). Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

$$\begin{array}{c|ccc}
& \text{Lower} & \text{Middle} & \text{Upper} \\
\hline
\text{Lower} & 0.7 & 0.2 & 0.1 \\
\text{Middle} & 0.2 & 0.6 & 0.2 \\
\text{Upper} & 0.1 & 0.4 & 0.5 \\
\end{array}$$

What fraction of families are upper class in the long run?
Solution. The transition probability matrix is regular, so a limiting distribution \( \pi = (\pi_{\text{Lower}}, \pi_{\text{Middle}}, \pi_{\text{Upper}}) \) exists and is the unique solution to the equations

\[
\begin{align*}
\pi_{\text{Lower}} &= 0.7\pi_{\text{Lower}} + 0.2\pi_{\text{Middle}} + 0.1\pi_{\text{Upper}}, \\
\pi_{\text{Middle}} &= 0.2\pi_{\text{Lower}} + 0.6\pi_{\text{Middle}} + 0.4\pi_{\text{Upper}}, \\
\pi_{\text{Upper}} &= 0.1\pi_{\text{Lower}} + 0.2\pi_{\text{Middle}} + 0.5\pi_{\text{Upper}}, \\
\pi_{\text{Lower}} + \pi_{\text{Middle}} + \pi_{\text{Upper}} &= 1.
\end{align*}
\]

Solving this system yields \( \pi = (6/17, 7/17, 4/17) \). Thus, the long run mean fraction of families that are upper class is \( \pi_{\text{Upper}} = 4/17 \). \( \square \)

Problem 3 (Pinsky & Karlin, Problem 4.1.2). Five balls are distributed between two urns, labeled A and B. Each period, one of the five balls is selected at random, and whichever urn it’s in, it is moved to the other urn. In the long run, what fraction of time is urn A empty?

Solution. For each nonnegative integer \( n \), let \( X_n \) be the number of balls in urn A at time \( n \). Then \( (X_n)_{n=0}^\infty \) is a time-homogeneous Markov chain with transition probability matrix

\[
P = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1/5 & 0 & 4/5 & 0 & 0 \\
1 & 0 & 2/5 & 0 & 3/5 & 0 \\
2 & 0 & 0 & 3/5 & 0 & 2/5 \\
3 & 0 & 0 & 0 & 4/5 & 0 \\
4 & 0 & 0 & 0 & 0 & 1/5 \\
5 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

This is because if there are \( i \) balls in urn A at time \( n \), then there with probability \( i/5 \), one of the \( i \) balls in urn A will be drawn and placed in urn B at time \( n + 1 \). Likewise, if there are \( i \) balls in urn A at time \( n \), then with probability \( (5 - i)/5 \), one of the \( 5 - i \) balls in urn B will be drawn and placed in urn A at time \( n + 1 \).

Inspection shows that the matrix \( P \) is not regular, so we cannot a priori apply the theory outlined in Section 4.1 of our textbook. However, we can still
compute the long-run mean fraction of time that the Markov chain spends in state 0. We wish to compute the limit as \( m \to \infty \) of

\[
E \left[ \frac{1}{m} \sum_{n=0}^{m-1} 1_{\{X_n=0\}} \mid X_0 = i \right] = \frac{1}{m} \sum_{n=0}^{m-1} E \left[ 1_{\{X_n=0\}} \mid X_0 = i \right] = \frac{1}{m} \sum_{n=0}^{m-1} P(X_n = 0 \mid X_0 = i) = \frac{1}{m} \sum_{n=0}^{m-1} P_{i,0}^{(n)}.
\]

We can check (by computing \( P^n \) using WolframAlpha) that

\[
\begin{align*}
P_{0,0}^{(n)} &= \frac{1}{32 \cdot 5^n} (10 + (-5)^n + 5 (-3)^n + 10 (-1)^n + 5 \cdot 3^n + 5^n) \\
P_{1,0}^{(n)} &= \frac{1}{32 \cdot 5^n} \left( 2 - (-5)^n - 2 (-1)^n + 3^{n+1} - (-1)^n 3^{n+1} + 5^n \right) \\
P_{2,0}^{(n)} &= \frac{1}{32 \cdot 5^n} (-2 + (-5)^n + (-3)^n - 2 (-1)^n + 3^n + 5^n) \\
P_{3,0}^{(n)} &= \frac{1}{32 \cdot 5^n} (-2 - (-5)^n + (-3)^n + 2 (-1)^n - 3^n + 5^n) \\
P_{4,0}^{(n)} &= \frac{1}{32 \cdot 5^n} \left( 2 + (-5)^n + 2 (-1)^n - 3^{n+1} - (-1)^n 3^{n+1} + 5^n \right) \\
P_{5,0}^{(n)} &= \frac{1}{32 \cdot 5^n} (10 - (-5)^n + 5 (-3)^n - 10 (-1)^n - 5 \cdot 3^n + 5^n)
\end{align*}
\]

Thus, for each \( i = 0, \ldots, 5 \) we have

\[
P_{i,0}^{(n)} = \frac{1}{32} + a_i (-1)^n + b_i \left( \frac{3}{5} \right)^n + c_i \left( -\frac{3}{5} \right)^n + d_i \left( \frac{1}{5} \right)^n + e_i \left( -\frac{1}{5} \right)^n
\]

for some \( a_i, b_i, c_i, d_i, e_i \in \mathbb{R} \) whose values don’t matter to us. Next, we use the fact that

\[
\sum_{n=0}^{m-1} r^n = \frac{1 - r^m}{1 - r}
\]

whenever \( r \neq 1 \), so

\[
\frac{1}{m} \sum_{n=0}^{m-1} P_{i,0}^{(n)} = \frac{1}{32} + \frac{a_i (1 - (-1)^m)}{2m} + \frac{b_i (1 - (3/5)^m)}{(2/5)m} + \frac{c_i (1 - (-3/5)^m)}{(8/5)m} + \frac{d_i (1 - (1/5)^m)}{(4/5)m} + \frac{e_i (1 - (-1/5)^m)}{(6/5)m},
\]

which converges to 1/32 as \( m \to \infty \). Therefore, the long-run mean fraction of time that the urn is empty is 1/32.
Problem 4 (Pinsky & Karlin, Problem 4.1.3). A Markov chain has the transition probability matrix

\[
P = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 1 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix},
\]

where \( \alpha_i \geq 0 \), \( i = 1, \ldots, 6 \) and \( \alpha_1 + \cdots + \alpha_6 = 1 \). Determine the limiting probability of being in state 0.

Solution. Note: to ensure that the transition probability matrix is regular, assume \( \alpha_i > 0 \) for \( i = 1, \ldots, 6 \). Since \( P \) is regular (it can be checked that \( P^6 \) has only positive entries), a limiting distribution \( \pi = (\pi_0, \ldots, \pi_5) \) exists and is the unique solution to the system of equations

\[
\begin{align*}
\pi_0 &= \alpha_1 \pi_0 + \pi_1, \quad \pi_1 = \alpha_2 \pi_0 + \pi_2, \\
\pi_2 &= \alpha_3 \pi_0 + \pi_3, \quad \pi_3 = \alpha_4 \pi_0 + \pi_4, \\
\pi_4 &= \alpha_5 \pi_0 + \pi_5, \quad \pi_5 = \alpha_6 \pi_0, \\
\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 &= 1.
\end{align*}
\]

Solving this system yields \( \pi_0 = \frac{1}{5-\alpha_6} \), which is the limiting probability of being in state 0. \( \Box \)

Problem 5 (Pinsky & Karlin, Exercise 4.2.5). From purchase to purchase, a particular customer switches brands among products A, B, and C according to a Markov chain whose transition probability matrix is

\[
P = \begin{pmatrix}
A & B & C \\
A & 0.6 & 0.2 & 0.2 \\
B & 0.1 & 0.7 & 0.2 \\
C & 0.1 & 0.1 & 0.8 \\
\end{pmatrix}
\]
In the long run, what fraction of time does this customer purchase brand A?

**Solution.** Since \( P \) is a regular transition probability matrix, a limiting distribution \( \pi = (\pi_A, \pi_A, \pi_C) \) exists and is the unique solution to the system of equations

\[
\begin{align*}
\pi_A &= 0.6\pi_A + 0.1\pi_B + 0.1\pi_C, \\
\pi_B &= 0.2\pi_A + 0.7\pi_B + 0.1\pi_C, \\
\pi_C &= 0.2\pi_A + 0.2\pi_B + 0.8\pi_C, \\
\pi_A + \pi_B + \pi_C &= 1.
\end{align*}
\]

Solving this system yields \( \pi_A = 1/5 \), which is the long-run fraction of time the customer purchases brand A. \( \square \)

**Problem 6** (Pinsky & Karlin, Problem 4.2.6). Consider a computer system that fails on a given day with probability \( p \) and remains “up” with probability \( q = 1 - p \). Suppose the repair time is a random variable \( N \) having the probability mass function \( p(k) = \beta (1 - \beta)^{k-1} \) for \( k = 1, 2, \ldots \), where \( 0 < \beta < 1 \). Let \( X_n = 1 \) if the computer is operating on day \( n \) and \( X_n = 0 \) if not. Show that \( \{X_n\} \) is a Markov chain with transition matrix

\[
\begin{pmatrix}
0 & 1 \\
0 & \alpha & \beta \\
1 & p & q
\end{pmatrix}
\]

and \( \alpha = 1 - \beta \). Determine the long run probability that the computer is operating in terms of \( \alpha, \beta, p, \) and \( q \)

**Solution.** If the system is currently operational, then it fails at the next time with probability \( p \) independently of the history of the system. Similarly, the repair time \( N \) is a geometric random variable, which satisfies the memoryless property, so the probability that the system will be fixed at the next time given that it is currently down does not depend on how long the system
has been down. This shows that \((X_n)_{n=0}^\infty\) is a Markov chain. The transition probabilities determined by

\[
P(X_{n+1} = 0 \mid X_n = 1) = p
\]

and

\[
P(X_{n+1} = 1 \mid X_n = 0) = P(N = 1) = \beta,
\]

which implies that \((X_n)_{n=0}^\infty\) has the given transition probability matrix.

Next, the transition probability matrix is regular, so there exists a limiting distribution \(\pi = (\pi_0, \pi_1)\), which is the unique solution to the system of equations

\[
\begin{align*}
\pi_0 &= \alpha \pi_0 + p \pi_1, \\
\pi_1 &= \beta \pi_0 + q \pi_1, \\
\pi_0 + \pi_1 &= 1
\end{align*}
\]

Solving this system yields \(\pi_1 = \beta/(\beta + p)\), which is the long-run probability that the computer is operational.

\boxed{
\textbf{Problem 7 (Pinsky & Karlin, Exercise 4.3.2). Which states are transient and which are recurrent in the Markov chain whose transition probability matrix is}

\[
P = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\
1 & 1/2 & 1/4 & 1/4 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 \\
3 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 \\
4 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\textbf{Solution.} Let \((X_n)_{n=0}^\infty\) denote the Markov chain with the given transition probability matrix. To simplify notation, let

\[
R_i = \{X_n = i \text{ for some } n \geq 1\}
\]

be the event that the Markov chain reaches state \(i\) at a time after the initial time, and let \(f_{i,i} = P(R_i \mid X_0 = i)\) be the probability that the Markov chain starts in state \(i\) and returns to state \(i\) at a later time. Then, by definition, state \(i\) is recurrent if and only if \(f_{i,i} = 1\). We treat each state one-by-one.
(i) State 0 is transient. \textbf{Proof:} By first-step analysis and the Markov property, we have
\begin{align*}
f_{0,0} &= P(R_0 \mid X_0 = 0, X_1 = 0)P(X_1 = 0 \mid X_0 = 0) \\
&\quad + P(R_0 \mid X_0 = 0, X_1 = 2)P(X_1 = 2 \mid X_0 = 0) \\
&\quad + P(R_0 \mid X_0 = 0, X_1 = 5)P(X_1 = 5 \mid X_0 = 0) \\
&= \frac{1}{3}P(R_0 \mid X_1 = 0) + \frac{1}{3}P(R_0 \mid X_1 = 2) + \frac{1}{3}P(R_0 \mid X_1 = 5) = \frac{1}{3},
\end{align*}

since \( P(R_0 \mid X_1 = 2) = P(R_0 \mid X_1 = 5) = 0 \) (because once the Markov chain enters state 2 or state 5, it will almost surely never return to state 0). Since \( f_{0,0} < 1 \), state 0 is transient.

(ii) State 1 is transient. \textbf{Proof:} The same argument as above shows that \( f_{1,1} = 1/4 \), so state 1 is transient.

(iii) State 2 is recurrent. \textbf{Proof:} Since
\begin{align*}
P(X_2 = 2 \mid X_0 = 2) &= P(X_2 = 2 \mid X_1 = 4)P(X_1 = 4 \mid X_0 = 2) = 1
\end{align*}

and \( \{X_2 = 2\} \subseteq R_2 \), it follows that \( f_{2,2} = P(R_2 \mid X_0 = 2) = 1 \), and hence state 2 is recurrent.

(iv) State 3 is transient. \textbf{Proof:} The transition probabilities of the Markov chain show that the process \textit{never} enters state 3 from another state, so clearly state 3 is transient.

(v) State 4 is recurrent. \textbf{Proof:} Since 2 \( \leftrightarrow \) 4 (i.e., states 2 and 4 communicate) and state 2 is recurrent, it follows that state 4 is recurrent.

(vi) State 5 is recurrent. \textbf{Proof:} Since \( P(X_1 = 5 \mid X_0 = 5) = 1 \) and \( \{X_1 = 5\} \subseteq R_5 \), it follows that \( f_{5,5} = P(R_5 \mid X_0 = 5) = 1 \), and hence state 5 is recurrent.

To summarize, states 2, 4, and 5 are recurrent, and the remaining states are transient.

\begin{Problem}
(Pinsky & Karlin, Problem 4.3.1). A two-state Markov chain has the transition probability matrix
\begin{equation*}
P = \begin{bmatrix}
0 & 1 \\
1 - a & a \\
1 & 1 - b
\end{bmatrix}
\end{equation*}
\end{Problem}
(a) Determine the first return distribution

\[ f^{(n)}_{0,0} = P(X_1 \neq 0, \ldots, X_{n-1} \neq 0, X_n = 0 \mid X_0 = 0). \]

(b) Verify equation (4.16) when \( i = 0 \):

\[ P(n)_{i,i} = \sum_{k=0}^{n} f^{(k)}_{i,i} P(n-k)_{i,i}, \quad n \geq 1. \] (4.16)

**Solution.** (a) By definition, \( f^{(0)}_{0,0} = 0 \). Moreover,

\[ f^{(1)}_{0,0} = P(X_1 = 0 \mid X_0 = 0) = 1 - a, \]

and, if \( n \geq 2 \), then by the Markov property it follows that

\[ f^{(n)}_{0,0} = P(X_1 = 1, \ldots, X_{n-1} = 1, X_n = 0 \mid X_0 = 0) \]

\[ = P(X_1 = 1 \mid X_0 = 0)P(X_n = 0 \mid X_{n-1} = 0) \prod_{k=1}^{n-2} P(X_{k+1} = 1 \mid X_k = 1) \]

\[ = ab(1 - b)^{n-2}. \]

(b) Note that equation (4.16) is proved in your textbook using first-step analysis, but we can verify it directly in the case of this problem. By equation (3.31) on page 112 in the textbook, we have

\[ P^{(n)}_{0,0} = \frac{b + a(1-a-b)^n}{a+b} \]

for all \( n \geq 1 \). Therefore, it suffices to show that

\[ \frac{b + a(1-a-b)^n}{a+b} = (1-a) \frac{b + a(1-a-b)^{n-1}}{a+b} \]

\[ + \sum_{k=2}^{n} ab(1-b)^{k-2} \frac{b + a(1-a-b)^{n-k}}{a+b}. \] (8.1)

We will evaluate the sum in equation 8.1 using the following geometric sum identity:

\[ \sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r} \]
whenever $r \neq 1$. Now, by pulling out constants, separating the sum into two, re-indexing (letting $j = k - 2$), using the geometric sum formula, and simplifying, we have

$$
\sum_{k=2}^{n} ab(1 - b)^{k-2} \frac{b + a (1 - a - b)^{n-k}}{a + b}
$$

$$
= \frac{ab}{a + b} \left( b \sum_{j=0}^{n-2} (1 - b)^j + a (1 - a - b)^{n-2} \sum_{j=0}^{n-2} \left( \frac{1 - b}{1 - a - b} \right)^j \right)
$$

$$
= \frac{ab}{a + b} \left( 1 - (1 - b)^{n-1} - (1 - a - b)^{n-1} \left( 1 - \left( \frac{1 - b}{1 - a - b} \right)^{n-1} \right) \right)
$$

$$
= \frac{ab}{a + b} \left( 1 - (1 - a - b)^{n-1} \right).
$$

Now this expression can be plugged into the right-hand side of equation (8.1), and some algebraic manipulation will show that the two sides are equal. \qed