INTRODUCTION TO STOCHASTIC PROCESSES.
MARKOV RANDOM FIELDS

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Recall that in doing Problem 1 on the first midterm we learned that the weather today depends upon the weather yesterday and tomorrow, even assuming the Markov property. Let's make a short excursion to generalize this idea, and in so doing, answer the question of what we might mean by a Markov property for a stochastic process whose index set is something other than (a subset of) the integers.

**Definition.** A graph $G = (V, E)$ consists of a set $V$ of vertices and a symmetric relation $E \subseteq V \times V$. Elements of $E$ are called edges; each is an unordered pair of vertices.

**Definition.** For any $v \in V$, let the neighborhood of $v$, $N(v) = \{w \in V \mid (v, w) \in E\}$, i.e., all vertices in $V$ that are connected to $v$ by an edge.

**Definition.** Given a graph $G = (V, E)$, let $\{X_v \mid v \in V\}$ be a stochastic process (with index set $V$). Suppose that for all $v \in V$,

$$\Pr(X_v = x_v \mid X_w = x_w, w \in V \setminus \{v\}) = \Pr(X_v = x_v \mid X_w = x_w, w \in N(v)).$$

Then the stochastic process is a Markov random field. We understand this as saying that the random variable $X_v$, conditioned on the values of the random variables at the neighboring vertices, is independent of the remaining random variables.

Now consider a very simple graph with $V = \mathbb{Z}_n$, the integers modulo $n \in \mathbb{N}$, so we can label them $V = \{0, 1, \ldots, n - 1\}$, and $E$ consisting of all the pairs $(i, i + 1)$ where $+$ is interpreted modulo $n$. In other words, $G$ is a ring of $n$ vertices. Define a stochastic process $\{X_i \in \{-1, +1\} \mid i \in \mathbb{Z}_n\}$, with joint probability function for $X = (X_0, \ldots, X_{n-1})$ is

$$\Pr(X = x) = \frac{1}{Z(\beta, n)} e^{-\beta \sum x_i x_{i+1}}.$$
where $\beta > 0$ and the partition function,

$$Z(\beta, n) = \sum_x e^{-\beta \sum_i x_i x_{i+1}}.$$  \hfill (*)

This is a one-dimensional Ising model with periodic boundary conditions. It is a model for magnetic materials: $\{-1, +1\}$ denote “spin down” and “spin up” for a magnetic moment, and $\beta$ is (proportional to) the inverse temperature.

EC2.a. Prove that this stochastic process is a Markov random field.

b. Compute $Z(\beta, n)$, i.e., simplify (*).

c. What is $\lim_{n \to \infty} Z(\beta, n)$?