Please simplify your answers to the extent reasonable without a calculator. Show your work. Explain your answers, concisely.

1. Consider the following game: A player rolls a die with the numbers 1, . . . , 6 on its faces. If it shows $k$, the player rolls it $k$ more times. The player wins if at least one of the last $k$ rolls shows a number greater than $k$.
   a. [20 points] What is $\Pr(\text{win}|\text{first roll shows } k)$?
   b. [10 points] What is $\Pr(\text{win})$?

2. [15 points] Suppose men’s heights are normally distributed with a mean of 70 inches and a standard deviation of 4 inches. Suppose, furthermore, that there is a positive correlation $\rho = 0.5$ between the height of a father and son. If William is 74 inches tall, what is the expected height of his son, Billy? If Alfred is 60 inches tall, what is the expected height of his son, Freddie?

3. Consider a game which involves flipping a coin: winning $1 when it lands head up and losing $1 when it lands tail up. Let $W_t \in \mathbb{Z}$ denote the number of dollars a player has after playing the game $t$ times, and let $X_t \in \{0, 1, 2\}$ be the reminder of $W_t$ upon division by 3. If $X_t = 0$ the player flips coin $B_0$ with $\Pr(\text{heads}) = 1/10$; otherwise the player flips coin $B_1$ with $\Pr(\text{heads}) = 3/4$.
   a. [10 points] Explain why $\{X_t | t \in \mathbb{N}\}$ is a Markov process.
   b. [10 points] Write the transition probability matrix for this discrete-time Markov chain.
   c. [6 points] Compute $\mathbb{E}[W_1|W_0 = 1]$, $\mathbb{E}[W_1|W_0 = 2]$ and $\mathbb{E}[W_1|W_0 = 3]$.
   d. [4 points] Is this stochastic process a Martingale? Why or why not?

4. Let $X_1$ and $X_2$ each take values in $\{0, 1\}$. Suppose the correlation $R = \text{Corr}[X_1, X_2]$ is a random variable chosen uniformly from the interval $(-1, 1)$ and $\Pr(X_i = 1|R = \rho) = 1/2$ for all $\rho \in (-1, 1)$.
   a. [10 points] What is $\Pr(X_1 = X_2|R = \rho)$?
   b. [10 points] What is $\Pr(X_1 = X_2)$?
   c. [5 points] Are $X_1$ and $X_2$ independent? Why or why not?

5. [Extra credit: 25 points] We have seen that Markov’s inequality is often not a very tight bound. Can you find a probability distribution function $f_X(x)$ for a continuous positive random variable $X$ such that Markov’s inequality is always saturated? (I.e., $\Pr(X \geq \lambda) = \mathbb{E}[X]/\lambda$ for all $\lambda > 0$. If not, prove that no such function exists.