Math 180B Homework 4 Solutions

Note: We will make repeated use of the following result.

**Lemma 1.** Let \((X_n)_{n=0}^\infty\) be a time-homogeneous Markov chain with countable state space \(S\), let \(A \subseteq S\), and let 
\[
T = \inf\{n \geq 0 : X_n \in A\}
\]
be the first time that the Markov chain reaches a state in \(A\). Then for each \(i \in S \setminus A\), we have 
\[
E[T | X_0 = i] = 1 + \sum_{j \in S \setminus A} P(X_1 = j | X_0 = i) E[T | X_0 = j].
\]

**Proof.** Suppose \(i \in S \setminus A\) and \(j \in S\). By the Markov property and time homogeneity, we have 
\[
E[T | X_0 = i, X_1 = j] = \sum_{n=0}^\infty P(T > n | X_0 = i, X_1 = j)
\]
\[
= \sum_{n=0}^\infty P(X_0 \notin A, \ldots, X_n \notin A | X_0 = i, X_1 = j)
\]
\[
= 1 + \sum_{n=1}^\infty P(X_1 \notin A, \ldots, X_n \notin A | X_0 = i, X_1 = j)
\]
\[
= 1 + \sum_{n=1}^\infty P(X_1 \notin A, \ldots, X_{n-1} \notin A | X_0 = j)
\]
\[
= 1 + \sum_{n=1}^\infty P(T \geq n | X_1 = j)
\]
\[
= 1 + E[T | X_0 = j].
\]

Also, note that if \(j \in A\), then \(E[T | X_0 = j] = 0\). Therefore, by first-step analysis, if \(i \notin A\), then 
\[
E[T | X_0 = i] = \sum_{n=0}^\infty n P(T = n | X_0 = i)
\]
Problem 1 (Pinsky & Karlin, Exercise 3.4.1). Find the mean time to reach state 3 starting from state 0 for the Markov chain whose transition probability matrix is

\[
P = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0.4 & 0.3 & 0.2 & 0.1 \\
0 & 0.7 & 0.2 & 0.1 \\
0 & 0 & 0.9 & 0.1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Solution. Let \( T \) be the time it takes for the Markov chain to reach state 3. That is, if \((X_n)_{n=0}^{\infty}\) is the Markov chain in question, then

\[ T = \inf \left\{ n \geq 0 : X_n = 3 \right\}. \]

We need to compute \( E[T \mid X_0 = 0] \). Let \( g_i = E[T \mid X_0 = i] \) for \( i \in \{0, 1, 2, 3\} \). By Lemma 1, we have the following linear system of equations:

\[
\begin{align*}
g_0 &= 1 + 0.4g_0 + 0.3g_1 + 0.2g_2, \\
g_1 &= 1 + 0.7g_1 + 0.2g_2, \\
g_2 &= 1 + 0.9g_2.
\end{align*}
\]

Solving this system, we get \( E[T \mid X_0 = 0] = g_0 = 10 \). \( \Box \)
**Problem 2** (Pinsky & Karlin, Exercise 3.4.4). A coin is tossed repeatedly until two successive heads appear. Find the mean number of tosses required.

**Hint:** Let $X_n$ be the cumulative number of successive heads. The state space is $0, 1, 2$, and the transition probability matrix is

$$
P = \begin{pmatrix}
0 & 1 & 2 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
1 & \frac{1}{2} & 0 \\
2 & 0 & 1
\end{pmatrix}.
$$

Determine the mean time to reach state 2 starting from state 0 by invoking a first step analysis.

**Solution.** Let

$$T = \inf \{ n \geq 0 : X_n = 2 \}.$$

We wish to compute $E[T \mid X_0 = 0]$. Let $g_i = E[T \mid X_0 = i]$ for $i \in \{0, 1, 2\}$. By Lemma 1, we get the following linear system of equations:

$$g_0 = 1 + \frac{1}{2}g_0 + \frac{1}{2}g_1, \quad g_1 = 1 + \frac{1}{2}g_0.$$

Therefore, $E[T \mid X_0 = 0] = g_0 = 6$. \qed

**Problem 3** (Pinsky & Karlin, Problem 3.4.1). Which will take fewer flips, on average: successively flipping a quarter until the pattern $HHT$ appears, i.e., until you observe two successive heads followed by a tails; or successively flipping a quarter until the pattern $HTH$ appears? Can you explain why these are different?

**Solution.** Let $(X_n)_{n=0}^\infty$ be the Markov chain whose states are the possible patterns of three consecutive coin tosses. Then $(X_n)_{n=0}^\infty$ has the following
### Transition Probability Matrix

The transition probability matrix (blank entries represent 0):

<table>
<thead>
<tr>
<th></th>
<th>$TTT$</th>
<th>$TTH$</th>
<th>$THT$</th>
<th>$THH$</th>
<th>$HTT$</th>
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<th>$HHT$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$TTT$</td>
<td>1/2</td>
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<tr>
<td>$TTH$</td>
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<td>1/2</td>
</tr>
</tbody>
</table>

### Define

$T_{HHT} = \inf\left\{ n \geq 0 : X_n = HHT \right\}$, $T_{HTH} = \inf\left\{ n \geq 0 : X_n = HTH \right\}$.

We wish to compare $E[T_{HHT}]$ and $E[T_{HTH}]$.

Let $g_i = E[T_{HHT} \mid X_0 = i]$ for each $i$. Then by Lemma 1, we have the following linear system of equations:

- $g_{TTT} = 1 + \frac{1}{2} g_{TTT} + \frac{1}{2} g_{TTH}$
- $g_{TTH} = 1 + \frac{1}{2} g_{THT} + \frac{1}{2} g_{THH}$
- $g_{THT} = 1 + \frac{1}{2} g_{HTT} + \frac{1}{2} g_{HTH}$
- $g_{THH} = 1 + \frac{1}{2} g_{HHT} + \frac{1}{2} g_{HHT}$
- $g_{HTT} = 1 + \frac{1}{2} g_{TTT} + \frac{1}{2} g_{TTH}$
- $g_{HHH} = 1 + \frac{1}{2} g_{HHH}$

Solving this system yields

$g_{HHH} = g_{THH} = 2$, $g_{HTH} = g_{TTH} = 6$, $g_{TTT} = g_{HTT} = g_{THT} = 8$,

and hence

$E[T_{HHT}] = \sum_j P(X_0 = j) g_j = \frac{1}{8} \left( 2 + 2 + 6 + 6 + 8 + 8 + 8 + 8 \right) = 5$.

Next, let $h_i = E[T_{HTH} \mid X_0 = i]$ for each $i$. Then by Lemma 1, we have the following linear system of equations:

- $h_{TTT} = 1 + \frac{1}{2} h_{TTT} + \frac{1}{2} h_{TTH}$
- $h_{TTH} = 1 + \frac{1}{2} h_{THT} + \frac{1}{2} h_{THH}$
- $h_{THT} = 1 + \frac{1}{2} h_{HTT} + \frac{1}{2} h_{HTH}$
- $h_{THH} = 1 + \frac{1}{2} h_{HHT} + \frac{1}{2} h_{HHT}$
- $h_{HTT} = 1 + \frac{1}{2} h_{TTT} + \frac{1}{2} h_{TTH}$
- $h_{HHH} = 1 + \frac{1}{2} h_{HHH}$
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\[ h_{TTH} = 1 + \frac{1}{2} h_{THT} + \frac{1}{2} h_{THH}, \]
\[ h_{THT} = 1 + \frac{1}{2} h_{HTT}, \quad h_{HHT} = 1 + \frac{1}{2} h_{HTT}, \]
\[ h_{THH} = 1 + \frac{1}{2} h_{HHT} + \frac{1}{2} h_{HHH}, \quad h_{HHH} = 1 + \frac{1}{2} h_{HHT} + \frac{1}{2} h_{HHH}. \]

Solving this system yields

\[ h_{THT} = h_{HHT} = 6, \quad h_{TTH} = h_{THH} = h_{HHH} = 8, \quad h_{HTT} = h_{TTT} = 10, \]

and hence

\[ E[T_{HTH}] = \sum_j P(X_0 = j) h_j = \frac{1}{8} \left( 6 + 6 + 8 + 8 + 8 + 10 + 10 \right) = 7. \]

Thus, on average, it will take fewer flips to observe HHT than HTH. \(\square\)

**Problem 4** (Pinsky & Karlin, Problem 3.4.2). A zero-seeking device operates as follows: If it is in state \(m\) at time \(n\), then at time \(n+1\), its position is uniformly distributed over the states \(0, 1, \ldots, m-1\). Find the expected time until the device first hits zero starting from state \(m\).

**Note:** This is a highly simplified model for an algorithm that seeks a maximum over a finite set of points.

**Solution.** This device can be modeled by a Markov chain \((X_n)_{n=0}^{\infty}\) taking values in the nonnegative integers with transition probabilities given by

\[ P(X_{n+1} = j \mid X_n = m) = \begin{cases} \frac{1}{m}, & \text{if } m > 0 \text{ and } 0 \leq j < m, \\ 1, & \text{if } m = j = 0, \\ 0, & \text{otherwise.} \end{cases} \]

Let

\[ T = \inf \{ n \geq 0 : X_n = 0 \}. \]

We wish to compute \(E[T \mid X_0 = m]\) for all \(m \geq 0\). If \(m = 0\), then clearly \(E[T \mid X_0 = m] = 0\). Suppose \(m > 0\). By Lemma 1, we have

\[ E[T \mid X_0 = m] = 1 + \sum_{j=1}^{\infty} P(X_1 = j \mid X_0 = m) E[T \mid X_0 = j] \]
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\[ = 1 + \frac{1}{m} \sum_{j=1}^{m-1} E[T \mid X_0 = j] \]

We will prove by strong induction that \( E[T \mid X_0 = m] = \sum_{k=1}^{m} \frac{1}{k} \), and the base case \( m = 1 \) is clear. Now suppose \( m > 1 \) and that \( E[T \mid X_0 = j] = \sum_{k=1}^{j} \frac{1}{k} \) when \( 1 \leq j < m \). Then

\[
E[T \mid X_0 = m] = 1 + \frac{1}{m} \sum_{j=1}^{m-1} E[T \mid X_0 = j]
= 1 + \frac{1}{m} \sum_{j=1}^{m-1} \frac{j}{k} = 1 + \frac{1}{m} \left( \sum_{k=1}^{m-1} \frac{1}{k} \sum_{j=k}^{m-1} \frac{1}{k} \right)
= 1 + \frac{1}{m} \sum_{k=1}^{m-1} \frac{m-k}{k} = 1 + \sum_{k=1}^{m-1} \frac{1}{k} \frac{m-1}{m} = \sum_{k=1}^{m} \frac{1}{k}. \quad \square
\]

**Problem 5** (Pinsky & Karlin, Problem 3.4.7). Let \( X_n \) be a Markov chain with transition probabilities \( P_{i,j} \). We are given a “discount factor” \( \beta \) with \( 0 < \beta < 1 \) and a cost function \( c(i) \), and we wish to determine the total expected discounted cost starting from state \( i \), defined by

\[
h_i = E \left[ \sum_{n=0}^{\infty} \beta^n c(X_n) \mid X_0 = i \right].
\]

Using a first step analysis show that \( h_i \) satisfies the system of linear equations

\[
h_i = c(i) + \beta \sum_j P_{i,j} h_j.
\]

**Solution.** Assuming \( c(i) \geq 0 \) for all \( i \) (so that we may interchange summation and expectation), we have

\[
h_i = E \left[ \sum_{n=0}^{\infty} \beta^n c(X_n) \mid X_0 = i \right]
= \sum_{n=0}^{\infty} \beta^n E[c(X_n) \mid X_0 = i]
= E[c(X_0) \mid X_0 = i] + \beta \sum_{n=1}^{\infty} \beta^{n-1} E[c(X_n) \mid X_0 = i]
\]
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\[ = c(i) + \beta \sum_{n=0}^{\infty} \beta^n E[c(X_{n+1}) \mid X_0 = i] \]

\[ = c(i) + \beta E \left[ \sum_{n=0}^{\infty} \beta^n c(X_{n+1}) \mid X_0 = i \right]. \]

Thus, it suffices to show that

\[ E \left[ \sum_{n=0}^{\infty} \beta^n c(X_{n+1}) \mid X_0 = i \right] = \sum_j P_{i,j} h_j. \] (5.1)

To see this, we use time-homogeneity and the Markov property:

\[ \sum_j P(X_1 = j \mid X_0 = i) E[c(X_n) \mid X_0 = j] \]

\[ = \sum_j P(X_1 = j \mid X_0 = i) \sum_k c(k) P(X_n = k \mid X_0 = j) \]

\[ = \sum_{j,k} c(k) P(X_1 = j \mid X_0 = i) P(X_{n+1} = k \mid X_1 = j) \]

\[ = \sum_{j,k} c(k) P(X_{n+1} = k, X_1 = j \mid X_0 = i) \]

\[ = \sum_k c(k) P(X_{n+1} = k \mid X_0 = i) \]

\[ = E[c(X_{n+1}) \mid X_0 = i], \]

and so

\[ \sum_j P_{i,j} h_j = \sum_j P(X_1 = j \mid X_0 = i) E \left[ \sum_{n=0}^{\infty} \beta^n c(X_n) \mid X_0 = j \right] \]

\[ = \sum_{n=0}^{\infty} \beta^n \sum_j P(X_1 = j \mid X_0 = i) E[c(X_n) \mid X_0 = j] \]

\[ = \sum_{n=0}^{\infty} \beta^n E[c(X_{n+1}) \mid X_0 = i] \]

\[ = E \left[ \sum_{n=0}^{\infty} \beta^n c(X_{n+1}) \mid X_0 = i \right]. \]

Therefore, (5.1) holds. \qed
Problem 6 (Pinsky & Karlin, Problem 3.4.8). An urn contains five red and three green balls. The balls are chosen at random, one by one, from the urn. If a red ball is chosen, it is removed. Any green ball that is chosen is returned to the urn. The selection process continues until all of the red balls have been removed from the urn. What is the mean duration of the game?

Solution. Let $X_0$ be the initial number of red balls in the urn, and for $n \geq 1$, let $X_n$ denote the number of red balls in the urn after the $n$th draw. Then $(X_n)_{n=0}^{\infty}$ is a Markov chain with transition matrix

$$
P = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1/4 & 3/4 & 0 & 0 & 0 \\
2 & 0 & 2/5 & 3/5 & 0 & 0 \\
3 & 0 & 0 & 3/6 & 3/6 & 0 \\
4 & 0 & 0 & 0 & 4/7 & 3/7 \\
5 & 0 & 0 & 0 & 0 & 5/8 \end{bmatrix}
$$

Let $T = \inf \{ n \geq 0 : X_n = 0 \}$ be the time until all red balls are removed from the urn and the game ends. We wish to compute $E[T \mid X_0 = 5]$. Let $g_i = E[T \mid X_0 = i]$. By Lemma 1, we have the following linear system of equations:

$$
\begin{align*}
g_1 &= 1 + \frac{3}{4} g_1, \\
g_2 &= 1 + \frac{2}{5} g_1 + \frac{3}{5} g_2, \\
g_3 &= 1 + \frac{3}{6} g_2 + \frac{3}{6} g_3, \\
g_4 &= 1 + \frac{4}{7} g_3 + \frac{3}{7} g_4, \\
g_5 &= 1 + \frac{5}{8} g_4 + \frac{3}{8} g_5.
\end{align*}
$$

Solving this system, we get $E[T \mid X_0 = 5] = g_5 = 277/20$. □
Problem 7 (Pinsky & Karlin, Problem 3.4.15). A simplified model for the spread of a rumor goes this way: There are $N = 5$ people in a group of friends, of which some have heard the rumor and the others have not. During any single period of time, two people are selected at random from the group and assumed to interact. The selection is such that an encounter between any pair of friends is just as likely as between any other pair. If one of these persons has heard the rumor and the other has not, then with probability $\alpha = 0.1$ the rumor is transmitted. Let $X_n$ denote the number of friends who have heard the rumor at the end of the $n$th period.

Assuming that the process begins at time 0 with a single person knowing the rumor, what is the mean time that it takes for everyone to hear it?

Solution. The stochastic process $(X_n)_{n=0}^{\infty}$ is a Markov chain with state space \{1, 2, 3, 4, 5\} and transition probabilities given by $P(X_{n+1} = 5 \mid X_n = 5) = 1$, and if $i \in \{1, 2, 3, 4\}$, then

$$P(X_{n+1} = i + 1 \mid X_n = i) = \alpha \cdot \binom{i}{1} \frac{(5-i)}{\binom{5}{2}} = \frac{i(5-i)}{100},$$

and

$$P(X_{n+1} = i \mid X_n = i) = 1 - \frac{i(5-i)}{100}.$$

All other transition probabilities are zero. Let $T = \inf \{n \geq 0 : X_n = 5\}$ be the time until all five people have heard the rumor. We wish to compute $E[T \mid X_0 = 1]$. Let $g_i = E[T \mid X_0 = i]$ for $i = 1, 2, 3, 4, 5$. By Lemma 1, we have the following linear system of equations:

$$
\begin{align*}
g_1 &= 1 + \left(1 - \frac{4}{100}\right)g_1 + \frac{4}{100}g_2, \\
g_2 &= 1 + \left(1 - \frac{2 \cdot 3}{100}\right)g_2 + \frac{2 \cdot 3}{100}g_3, \\
g_3 &= 1 + \left(1 - \frac{3 \cdot 2}{100}\right)g_3 + \frac{2 \cdot 3}{100}g_4, \\
g_4 &= 1 + \left(1 - \frac{4}{100}\right)g_4.
\end{align*}
$$

Solving this system, we get $E[T \mid X_0 = 1] = g_1 = 250/3$. \qed
Ex 3.5.4 The transition probability matrix of the Markov chain (of states 0, 1, 2, 3) is

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

which is of the form of a success runs Markov chain with all probabilities \( p_i = q_i = \frac{1}{2} \).

Ex 3.5.8 The state space of the Markov chain is the nonnegative integers. The transition probabilities are as follows:

\[
P(X_{n+1} = k|X_n = j) = \begin{cases}
1 - \alpha & j = 0 = k \\
\alpha & j = 0, k = 1 \\
(1 - \alpha)\beta & j > 0, k = j - 1 \\
\alpha\beta + (1 - \alpha)(1 - \beta) & j = k > 0 \\
\alpha(1 - \beta) & j > 0, k = j + 1 \\
0 & \text{otherwise}
\end{cases}
\]

So corresponding to (3.38), \( r_0 = 1 - \alpha, p_0 = \alpha, q_i = (1 - \alpha)\beta, r_i = \alpha\beta + (1 - \alpha)(1 - \beta), p_i = \alpha(1 - \beta) \) for \( i \geq 1 \).

Pr 3.5.1 We are taking \( M = 3 \) in equation (3.35) in section 3.5.2, with \( P(\xi = i) = a_i \) for \( i = 0, 1, 2, 3 \) and \( a_0 + a_1 + a_2 + a_3 = 1 \). All the \( a_i = 0 \) with \( i \geq 4 \). Putting the corresponding \( a_3, a_4, \ldots \) into equation (3.35), we have \( \mu = 1/a_3 \). Since in this case, \( T = \min\{n \geq 1 : X_n \geq 3\} = \min\{n \geq 1 : X_n = 3\} \), \( \mu = \nu_0 \), the mean time until absorption starting from 0.

Pr 3.5.4 We let \( x_i, i = 1, 2, \ldots \) to be i.i.d. discrete random variable have uniform distribution on the integers 1, 2, \ldots, 6. \( X_0 = 0 \) and \( X_{n+1} = X_n + \xi_{n+1} \) is the Markov chain we are interested in. Set \( T = \min\{n \geq 0 : X_n > 10\} \). We wil solve \( u_i = P(X_T = 13|X_0 = i) \) for \( 0 \leq i \leq 10 \).

Clearly,

\[
u_i = \sum_{k=1}^{6} P(X_T = 13|X_i + 1 = i + k) P(\xi_{i+1} = k) = \sum_{k=1}^{6} \frac{u_{i+k}}{6}.
\]
So we have

\[
\begin{align*}
  u_{10} &= \frac{1}{6} \\
  u_9 &= \frac{1}{6} u_{10} + \frac{1}{6} \\
  u_8 &= \frac{1}{6} u_9 + \frac{1}{6} u_{10} + \frac{1}{6} \\
  u_7 &= \frac{1}{6} u_8 + \frac{1}{6} u_9 + \frac{1}{6} u_{10} + \frac{1}{6} \\
  u_6 &= \frac{1}{6} u_7 + \frac{1}{6} u_8 + \frac{1}{6} u_9 + \frac{1}{6} u_{10} \\
  u_5 &= \frac{1}{6} u_6 + \frac{1}{6} u_7 + \frac{1}{6} u_8 + \frac{1}{6} u_9 + \frac{1}{6} u_{10} \\
  u_4 &= \frac{1}{6} u_5 + \frac{1}{6} u_6 + \frac{1}{6} u_7 + \frac{1}{6} u_8 + \frac{1}{6} u_9 + \frac{1}{6} u_{10} \\
  u_3 &= \frac{1}{6} u_4 + \frac{1}{6} u_5 + \frac{1}{6} u_6 + \frac{1}{6} u_7 + \frac{1}{6} u_8 + \frac{1}{6} u_9 \\
  u_2 &= \frac{1}{6} u_3 + \frac{1}{6} u_4 + \frac{1}{6} u_5 + \frac{1}{6} u_6 + \frac{1}{6} u_7 + \frac{1}{6} u_8 \\
  u_1 &= \frac{1}{6} u_2 + \frac{1}{6} u_3 + \frac{1}{6} u_4 + \frac{1}{6} u_5 + \frac{1}{6} u_6 + \frac{1}{6} u_7 \\
  u_0 &= \frac{1}{6} u_1 + \frac{1}{6} u_2 + \frac{1}{6} u_3 + \frac{1}{6} u_4 + \frac{1}{6} u_5 + \frac{1}{6} u_6.
\end{align*}
\]

Solving the system of linear equations yields \( u_0 = \frac{65990113}{362797056} \).

Pr 3.5.5 (a) \( X_n \) is clearly nonnegative. We observe that for \( k \geq 1 \),

\[
\mathbb{E}[X_{n+1}|X_n = k] = \frac{1}{2}(k-1) + \frac{1}{2}(k+1) = k
\]

and

\[
\mathbb{E}[X_{n+1}|X_n = 0] = 0.
\]

Thus \( \mathbb{E}[X_{n+1}|X_n] = X_n \) and since

\[
\mathbb{E}[X_{n+1}] = \sum_{k=0}^{\infty} \mathbb{E}[X_{n+1}|X_n = k] \mathbb{P}(X_n = k) = \sum_{k=1}^{\infty} k \mathbb{P}(X_n = k) = \mathbb{E}[X_n],
\]

we have \( \mathbb{E}[|X_n|] < \infty \) for all \( n \).

(b) Using the maximal inequality,

\[
\mathbb{P}(\max_n X_n \geq N) \leq \frac{\mathbb{E}[X_0]}{N}
\]

if \( X_0 < N \), and

\[
\mathbb{P}(\max_n X_n \geq N) = 1
\]

if \( X_0 \geq N \).