(a) We have a Markov chain $X_0 = 3$, 

\[ P(X_{n+1} = k | X_n = j) = \begin{cases} 
\frac{1}{2} & k = j \pm 1, j \neq 0, 5 \\
1 & j = k, j = 0, 5 \\
0 & \text{otherwise}
\end{cases} \]
We have the stopping time \( T = \min\{n \geq 0 : X_n = 0 \text{ or } 5\} \) and want to compute \( \mathbb{P}(X_T = 5|X_0 = 3) \). Let \( u_i = \mathbb{P}(X_T = 5|X_0 = i) \), \( i = 0, 1, 2, 3, 4, 5 \). Clearly \( u_0 = 0 \), \( u_5 = 1 \). Using the first step analysis, we get

\[
\begin{align*}
    u_4 &= \frac{1}{2} u_3 + \frac{1}{2} u_5 = \frac{1}{2} u_3 + \frac{1}{2} \\
    u_3 &= \frac{1}{2} u_2 + \frac{1}{2} u_4 \\
    u_2 &= \frac{1}{2} u_1 + \frac{1}{2} u_3 \\
    u_1 &= \frac{1}{2} u_0 + \frac{1}{2} u_2 = \frac{1}{2} u_2.
\end{align*}
\]

So, \( u_3 = \frac{3}{2} u_2 \) and \( u_4 = \frac{4}{3} u_3 \). Finally the first equation tells us

\[
\left( \frac{4}{3} - \frac{1}{2} \right) u_3 = \frac{1}{2}
\]

i.e. \( u_3 = \frac{2}{5} \).

(b) We change the equations from part (a) to

\[
\begin{align*}
    u_4 &= (1 - p)u_3 + pu_5 = (1 - p)u_3 + p \\
    u_3 &= (1 - p)u_2 + pu_4 \\
    u_2 &= (1 - p)u_1 + pu_3 \\
    u_1 &= (1 - p)u_0 + pu_2 = pu_2.
\end{align*}
\]

A similar computation leads to

\[
u_3 = \frac{p^2(1 - p(1 - p))}{1 - 3p(1 - p) + p^2(1 - p)^2}.
\]
Exercise 3.8.1

Solution. The offspring distribution $\xi$ is given by $\Pr\{\xi = 0\} = 1/2$ and $\Pr\{\xi = 2\} = 1/2$. The mean and variance are $E[\xi] = 1$ and $\text{Var}[\xi] = 1/2 \cdot 0^2 + 1/2 \cdot 2^2 = 1$. The formulas (3.98, 3.99) for the mean and variance of a branching process (assuming $X_0 = 1$) apply:

$$E[X_n] = E[\xi]^n = 1$$
$$\text{Var}[X_n] = \text{Var}[\xi]E[\xi]^{n-1}n = n.$$

Exercise 3.8.4

Solution. This time the branching distribution is $\xi \sim \text{Poisson}(\lambda)$ and so $E[\xi] = \text{Var}[\xi] = \lambda$. As before we may apply the general formulas 3.98, 3.99 to find

$$E[X_n] = \lambda^n$$
$$\text{Var}[X_n] = \lambda \cdot \lambda^{n-1} = \begin{cases} n & \lambda = 1 \\ \frac{n \lambda^n}{1-\lambda} & \lambda \neq 1 \end{cases}.$$

Problem 3. (Problem 3.8.2) Small typo in book corrected: Let $Z = \sum_{n=0}^{\infty} X_n$ be the total family size in a branching process whose offspring distribution has a mean $\mu = E[\xi] < 1$. Assuming that $X_0 = 1$, show that $E[Z] = 1/(1-\mu)$.

Solution. We compute $E[Z]$ using the linearity of expectation:

$$E[Z] = \sum_{n=0}^{\infty} E[X_n].$$

We know $E[X_n] = \mu^n$ from (3.98) so

$$E[Z] = \sum_{n=0}^{\infty} \mu^n = \frac{1}{1 - \mu}$$

where the last equality is true because $\mu < 1$.

An alternate solution involves first step analysis. The first time step always contributes 1 to the expected total family size. In the remaining steps, we expect $E[Z]$-many additional members per offspring produced in the first step. The expected number of offspring produced in the first step is $\mu$, so altogether we have

$$E[Z] = 1 + \mu E[Z].$$

For $\mu \geq 1$ this is solved only by $E[Z] = \infty$ (since $E[Z] \geq 0$), but for $0 \leq \mu < 1$ we have the solution $E[Z] = \frac{1}{1-\mu}$. 

1
Solution. (a) Let $X_T$ denote the total number of members in a family. Let $X_k$ be the sex of the $k$th offspring. We write $\Pr(BG)$ for $\Pr(X_1 = B$ and $X_2 = G)$ (and similarly for other combinations). We also assume that the chances of having a male or female offspring are equal. Following the definition of the breeding mechanism, we deduce

\[
\begin{align*}
\Pr(X_T = 0) &= 0 \\
\Pr(X_T = 1) &= 0 \\
\Pr(X_T = 2) &= \Pr(BG) + \Pr(GB) + \Pr(GG) = \frac{3}{4} \\
\Pr(X_T = 3) &= \Pr(BBGG) = \frac{1}{8} \\
\Pr(X_T = 4) &= \Pr(BBBG) = \frac{1}{16}
\end{align*}
\]

\[\vdots\]

\[
\Pr(X_T = k) = \Pr(BB\ldots BG) = 2^{-k} \quad \text{for all } k > 2.
\]

(b) Let $X_B$ denote the total number of male children in a single family. Again we directly apply the definition of the breeding mechanism to find

\[
\begin{align*}
\Pr(X_B = 0) &= \Pr(GG) = \frac{1}{4} \\
\Pr(X_B = 1) &= \Pr(BG) + \Pr(BG) = \frac{1}{2} \\
\Pr(X_B = 2) &= \Pr(BBGG) = \frac{1}{8} \\
\Pr(X_B = 3) &= \Pr(BBBG) = \frac{1}{16}
\end{align*}
\]

\[\vdots\]

\[
\Pr(X_B = k) = \Pr(BB\ldots BG) = 2^{-k-1} \quad \text{for all } k > 1.
\]