Problem 7 (Pinsky & Karlin, Exercise 4.3.2). Which states are transient and which are recurrent in the Markov chain whose transition probability matrix is

\[
P = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\
1 & 1/2 & 1/4 & 1/4 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 \\
3 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 \\
4 & 0 & 1 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Solution. Let \((X_n)_{n=0}^\infty\) denote the Markov chain with the given transition probability matrix. To simplify notation, let

\[
R_i = \{X_n = i \text{ for some } n \geq 1\}
\]

be the event that the Markov chain reaches state \(i\) at a time after the initial time, and let \(f_{i,i} = P(R_i \mid X_0 = i)\) be the probability that the Markov chain starts in state \(i\) and returns to state \(i\) at a later time. Then, by definition, state \(i\) is recurrent if and only if \(f_{i,i} = 1\). We treat each state one-by-one.
(i) State 0 is transient. **Proof:** By first-step analysis and the Markov property, we have

\[
f_{0,0} = P(R_0 \mid X_0 = 0, X_1 = 0)P(X_1 = 0 \mid X_0 = 0) \\
+ P(R_0 \mid X_0 = 0, X_1 = 2)P(X_1 = 2 \mid X_0 = 0) \\
+ P(R_0 \mid X_0 = 0, X_1 = 5)P(X_1 = 5 \mid X_0 = 0)
\]

\[
= \frac{1}{3}P(R_0 \mid X_1 = 0) + \frac{1}{3}P(R_0 \mid X_1 = 2) + \frac{1}{3}P(R_0 \mid X_1 = 5) = \frac{1}{3},
\]

since \(P(R_0 \mid X_1 = 2) = P(R_0 \mid X_1 = 5) = 0\) (because once the Markov chain enters state 2 or state 5, it will almost surely never return to state 0). Since \(f_{0,0} < 1\), state 0 is transient.

(ii) State 1 is transient. **Proof:** The same argument as above shows that \(f_{1,1} = 1/4\), so state 1 is transient.

(iii) State 2 is recurrent. **Proof:** Since

\[
P(X_2 = 2 \mid X_0 = 2) = P(X_2 = 2 \mid X_1 = 4)P(X_1 = 4 \mid X_0 = 2) = 1
\]

and \(\{X_2 = 2\} \subseteq R_2\), it follows that \(f_{2,2} = P(R_2 \mid X_0 = 2) = 1\), and hence state 2 is recurrent.

(iv) State 3 is transient. **Proof:** The transition probabilities of the Markov chain show that the process never enters state 3 from another state, so clearly state 3 is transient.

(v) State 4 is recurrent. **Proof:** Since \(2 \leftrightarrow 4\) (i.e., states 2 and 4 communicate) and state 2 is recurrent, it follows that state 4 is recurrent.

(vi) State 5 is recurrent. **Proof:** Since \(P(X_1 = 5 \mid X_0 = 5) = 1\) and \(\{X_1 = 5\} \subseteq R_5\), it follows that \(f_{5,5} = P(R_5 \mid X_0 = 5) = 1\), and hence state 5 is recurrent.

To summarize, states 2, 4, and 5 are recurrent, and the remaining states are transient. \(\square\)

**Problem 8** (Pinsky & Karlin, Problem 4.3.1). A two-state Markov chain has the transition probability matrix

\[
P = \begin{pmatrix}
0 & 1 \\
1 - a & a \\
b & 1 - b
\end{pmatrix}
\]
(a) Determine the first return distribution

\[ f_{0,0}^{(n)} = P(X_1 \neq 0, \ldots, X_{n-1} \neq 0, X_n = 0 \mid X_0 = 0). \]

(b) Verify equation (4.16) when \( i = 0 \):

\[ P_{i,i}^{(n)} = \sum_{k=0}^{n} f_{i,i}^{(k)} P_{i,i}^{(n-k)}, \quad n \geq 1. \quad (4.16) \]

**Solution.** (a) By definition, \( f_{0,0}^{(0)} = 0 \). Moreover,

\[ f_{0,0}^{(1)} = P(X_1 = 0 \mid X_0 = 0) = 1 - a, \]

and, if \( n \geq 2 \), then by the Markov property it follows that

\[ f_{0,0}^{(n)} = P(X_1 = 1, \ldots, X_{n-1} = 1, X_n = 0 \mid X_0 = 0) \]

\[ = P(X_1 = 1 \mid X_0 = 0)P(X_n = 0 \mid X_{n-1} = 0) \prod_{k=1}^{n-2} P(X_{k+1} = 1 \mid X_k = 1) \]

\[ = ab (1 - b)^{n-2}. \]

(b) Note that equation (4.16) is proved in your textbook using first-step analysis, but we can verify it directly in the case of this problem. By equation (3.31) on page 112 in the textbook, we have

\[ P_{0,0}^{(n)} = \frac{b + a (1 - a - b)^n}{a + b} \]

for all \( n \geq 1 \). Therefore, it suffices to show that

\[ \frac{b + a (1 - a - b)^n}{a + b} = (1 - a) \frac{b + a (1 - a - b)^{n-1}}{a + b} \]

\[ + \sum_{k=2}^{n} ab (1 - b)^{k-2} \frac{b + a (1 - a - b)^{n-k}}{a + b}. \quad (8.1) \]

We will evaluate the sum in equation 8.1 using the following geometric sum identity:

\[ \sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r} \]
whenever \( r \neq 1 \). Now, by pulling out constants, separating the sum into two, re-indexing (letting \( j = k - 2 \)), using the geometric sum formula, and simplifying, we have

\[
\sum_{k=2}^{n} ab (1 - b)^{k-2} \frac{b + a \left(1 - a - b\right)^{n-k}}{a + b}
\]

\[
= \frac{ab}{a + b} \left( b \sum_{j=0}^{n-2} (1 - b)^j + a \left(1 - a - b\right)^{n-2} \sum_{j=0}^{n-2} \left( \frac{1 - b}{1 - a - b}\right)^j \right)
\]

\[
= \frac{ab}{a + b} \left(1 - (1 - b)^{n-1} - (1 - a - b)^{n-1} \left( 1 - \left( \frac{1 - b}{1 - a - b}\right)^{n-1} \right)\right)
\]

\[
= \frac{ab}{a + b} \left(1 - (1 - a - b)^{n-1}\right).
\]

Now this expression can be plugged into the right-hand side of equation (8.1), and some algebraic manipulation will show that the two sides are equal. \( \square \)
Ex. 4.4.3 Recall that the stationary distribution \( \pi \) is defined as the solution to the linear system:

\[
\pi = \pi P,
\]

\[
\frac{1}{\sum_{i=1}^{10} \pi_i} = 1.
\]

Therefore,

\[
\begin{align*}
\frac{1}{4} \pi_1 + \frac{1}{2} \pi_4 &= \pi_1, \\
\frac{1}{2} \pi_1 + \frac{1}{3} \pi_3 &= \pi_2, \\
\frac{1}{4} \pi_2 + \frac{1}{2} \pi_4 &= \pi_3, \\
\frac{1}{2} \pi_1 + \frac{2}{3} \pi_3 &= \pi_4, \\
\pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1
\end{align*}
\]

Subtracting (1) from (2), (2) from (4), we get

\[
\begin{align*}
\pi_1 &= 2\pi_3 - 2\pi_1, \\
\pi_3 &= 3\pi_4 - 3\pi_2
\end{align*}
\]

Plug (6) back into (2), (7) into (4), we will have

\[
\begin{align*}
3\pi_1 &= 2\pi_3, \\
3\pi_2 &= 2\pi_4
\end{align*}
\]

Plug them back in (5) the answer will be

\[
\begin{align*}
\pi_1 &= \frac{1}{10} \\
\pi_2 &= \frac{1}{10} \\
\pi_3 &= \frac{3}{10} \\
\pi_4 &= \frac{3}{10}
\end{align*}
\]
Pr 4.7 (I'll derive the answer in a different way, although it's a bit more different and has the same answer, it makes more sense in this way.)

I'll define the state space S as

\[ S = \{ \text{MRC, MRN, MSC, MSN} \} \]

where
- \text{MRC: morning, rainy, can stay in the same place}
- \text{MRN: not same}
- \text{MSC: sunny, same place}
- \text{MSN: sunny, not same}

Thus we'll get the one step transition matrix

\[
\begin{pmatrix}
p^2 & p(1-p) & p(1-p) & (1-p)^2 \\
p^2 & p(1-p) & p(1-p) & (1-p)^2 \\
p & 0 & 1-p & 0 \\
p^2 & p(1-p) & p(1-p) & (1-p)^2
\end{pmatrix}
\]

It is easy to show that the matrix is regular.

Solving the linear system

\[
\sum_T \lambda_T = 1 \\
\lambda_{\text{MRC}} + \lambda_{\text{MRN}} + \lambda_{\text{MSC}} + \lambda_{\text{MSN}} = 1
\]

gives us the limiting distribution \( T_C \).
Especially, we have \( T_{MRN} = \frac{p(1-p)}{2-p} \),
which, in the long run, denotes the fractions
of mornings you walk in the rain.

Now, the answer we're looking for is

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} E[I_i \text{ walk in the rain on the } i\text{th day}]
\]

\[
= \lim_{m \to \infty} \frac{1}{m} \sum P(\text{walk, rain, ith day})
\]

\[
= \lim_{m \to \infty} \frac{1}{m} \sum P(\text{walk rain ith day})
\]

\[
= \lim_{m \to \infty} \frac{1}{m} \sum [P(A_i) + P(B_i) - P(A_i \cap B_i)]
\]

By symmetry and the fact that \( P(A_i \cap B_i) \equiv 0 \),
we have

\[
\lim_{m \to \infty} \frac{1}{m} \sum P(A_i)
\]

\[
= 2 \pi_{MRN} = \frac{2p(1-p)}{2-p}
\]