1. Consider a game which involves flipping a coin: winning $1 when it lands head up and losing $1 when it lands tail up. Let $W_t \in \mathbb{Z}$ denote the number of dollars a player has after playing the game $t$ times, and let $X_t \in \{0, 1, 2\}$ be the reminder of $W_t$ upon division by 3. If $X_t = 0$ the player flips coin $B_0$ with Pr(heads) = $1/10$; otherwise the player flips coin $B_1$ with Pr(heads) = $3/4$.

a. [10 points] Explain why $\{X_t \mid t \in \mathbb{N}\}$ is a Markov process.

Pr($X_{t+1} = x_{t+1} | X_0 = x_0, \ldots, X_t = x_t$) = Pr($X_{t+1} = x_t$) since $x_t$ determines which coin is flipped.

b. [10 points] Write the transition probability matrix for this discrete-time Markov chain.

$$
P = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1/10 & 9/10 \\ 1/4 & 0 & 3/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}
$$

c. [6 points] Compute $E[W_1 | W_0 = 1]$, $E[W_1 | W_0 = 2]$ and $E[W_1 | W_0 = 3]$.

$$
E[W_1 | W_0 = 1] = \frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 0 = \frac{3}{2}
$$

$$
E[W_1 | W_0 = 2] = \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 1 = \frac{5}{2}
$$

$$
E[W_1 | W_0 = 3] = \frac{1}{10} \cdot 4 + \frac{9}{10} \cdot 2 = \frac{11}{5}
$$

d. [4 points] Is this stochastic process a Martingale? Why or why not?

No, because, for example, $E[X_1 | X_0 = 1] = E[W_1 | W_0 = 1] = \frac{3}{2} \neq 1$.

2. Especially in San Diego, a good guess about tomorrow’s weather is that it will be like today’s. Let’s formalize this as there being two weather states, $W_t \in \{\text{sunny, rainy}\}$, where the $t \in \mathbb{N}$ index labels days, and suppose they form a Markov chain with transition probability matrix:

$$
P = \begin{pmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{pmatrix},
$$

where the rows and columns are labeled by the states ‘sunny’ and ‘rainy’, in that order.

a. [5 points] What is Pr($W_{t+1} = \text{rainy} \mid W_t = \text{rainy}$)?

0.6, from $P$.

b. [10 points] What is Pr($W_{t+1} = \text{rainy} \mid W_t = \text{rainy}$ and $W_{t+2} = \text{rainy}$)?
Write \( R \) for ‘rainy’ and \( S \) for ‘sunny’. Then

\[
\Pr(W_{t+1} = R \mid W_t = R, W_{t+2} = R) = \frac{\Pr(W_{t+1} = R, W_{t+2} = R \mid W_t = R)}{\Pr(W_{t+2} = R \mid W_t = R)}
\]

\[
= \frac{\Pr(W_{t+1} = R, W_{t+2} = R \mid W_t = R)}{0.6 \cdot 0.6 + 0.4 \cdot 0.1} = 0.9.
\]

c. [10 points] What is \( \Pr(W_{t+1} = \text{rainy} \mid W_t = \text{rainy} \text{ and } W_{t+2} = \text{sunny}) \)?

By the same argument as in 2.b,

\[
\Pr(W_{t+1} = R \mid W_t = R, W_{t+2} = S) = \frac{0.6 \cdot 0.4}{0.6 \cdot 0.4 + 0.4 \cdot 0.9} = 0.4,
\]

so it is more likely to be sunny on a day between a rainy day and a sunny day.

3. [Extra credit: 15 points] Using the same information as in Problem 1, suppose there is a sequence of 16 days for which you know the weather on 5: \( S *** S *** R ** R *** * R \), where \( S \) denotes ‘sunny’, \( R \) denotes ‘rainy’, and \( * \) indicates we don’t know what the weather is. Which is the most probable sequence of \( S \)s and \( R \)s? Why?

Because of the Markov property we can consider each sequence of \( * \)s surrounded by known weather days separately. The probability of any sequence is given by the product of the appropriate entries in \( P \) from 1. Since 0.9 is the biggest entry, between \( S \)s the highest probability sequence is just \( S \)s. Since the transition from \( R \) to \( R \) has probability \( 0.6 < 0.9 \), the highest probability sequence between an \( S \) and an \( R \) is also all \( S \)s. Finally, between two \( R \)s the highest probability sequence is all \( R \)s since \( 0.4 < 0.6 \), unless the sequence is long enough that it is more probable to switch to a sequence of \( S \)s. That is, \( RR^n R \) has probability \( 0.6^{n+1} \), while \( RS^n R \) has probability \( 0.4 \cdot 0.9^{n-1} \cdot 0.1 \). The latter is larger for \( n \geq 7 \), but the numbers of \( * \)s in the last two surrounded sequences are 2 and 4, respectively. So the most probable sequence is \( SS S S S R R R R R R R R R R R R R R \).

In the somewhat more complicated setting of hidden Markov models, the Viterbi algorithm answers the analogous question. Some of you might find this worth learning about; Viterbi is one of the co-founders of Qualcomm.

4. Qingwa the frog starts at 0 on the number line. Each minute she hops one unit in the positive direction with probability \( 1/2 \) and stays still otherwise. Let \( X_n \) denote her position after \( n \) minutes.

a. [5 points] What is the transition probability matrix \( P \) for this Markov chain?

\[
P = \begin{pmatrix}
0 & 1 & 2 & 3 & \cdots \\
1/2 & 1/2 & & & \\
1/2 & 1/2 & 1/2 & & \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}.
\]
b. [10 points] What is the 2-step transition probability matrix $P^{(2)}$?

$$P^{(2)} = P^2 = \frac{1}{4} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \\ \ddots & \ddots & \ddots \end{pmatrix}.$$ 

c. [10 points] What is the probability that Qingwa’s position $X_n = k$ after $n$ minutes? These are the entries in the top row of $P^{(n)} = P^n$, which are

$$(P^n)_{0k} = \frac{1}{2^n} \binom{n}{k},$$

since each sequence of $n$ hops/no hops has the same probability $1/2^n$ and there are $\binom{n}{k}$ of those with $k$ hops.

5. Two bugs start at opposite corners of a square. During each minute they each walk along an edge, chosen uniformly at random, to an adjacent corner. If they run into each other on an edge they stop, and if they end at the same corner, they stop. Let $T$ be the time at which they stop.

a. [15 points] Describe this situation as a discrete time Markov chain: specify a set of states for the system and write down the one step probability transition matrix. Let $X_t$ = how far apart the bugs are at time $t$, counted by numbers of edges one must traverse to reach the other $\in \{0, 1, 2\}$. Then

$$P = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$ 

b. [10 points] What is $E[T]$?

Let $\nu_1 = E[T|X_0 = i]$. Then we have

$$\nu_0 = 0,$$

$$\nu_1 = 1 + \nu_1 \cdot \frac{3}{4} \quad \Rightarrow \quad \nu_1 = 4,$$

$$\nu_2 = 1 + \nu_2 \cdot \frac{1}{2} \quad \Rightarrow \quad \nu_2 = 2,$$

so since the bugs start at opposite corners, $E[T] = 2$.

c. [10 points] What would $E[T]$ be if the bugs start on adjacent vertices?

As calculated above, in this case, $E[T] = \nu_1 = 4$. 
6. Suppose when people get the flu and don’t stay home in bed they infect 2 other people with probability \( p > 1/2 \) and infect no one with probability \( 1 - p \).

a. [15 points] What is the probability, starting with a single person with the flu, that the flu epidemic will die out?

Setting the extinction probability \( u \) equal to the probability generating function gives

\[
  u = 1 - p + pu^2 \quad \Rightarrow \quad 0 = pu^2 - u + 1 - p = (u - 1)(pu - (1 - p))
\]

\[
  \Rightarrow \quad u = \frac{1 - p}{p} < 1 \quad \text{for} \quad p > \frac{1}{2}.
\]

b. [20 points] If we could change people’s behavior so that when they get sick, they stay home in bed with probability \( q > 0 \). How big must \( q \) be (in terms of \( p \)) to make the extinction probability for the flu be 1?

In this case the extinction probability satisfies

\[
  u = q + (1 - q)(1 - p) + (1 - q)pu^2 \quad \Rightarrow \quad 0 = (1 - q)pu^2 - u + q + (1 - q)(1 - p)
\]

\[
  = (u - 1)((1 - q)pu - [q + (1 - q)(1 - p)])
\]

\[
  \Rightarrow \quad u = \frac{q + (1 - q)(1 - p)}{(1 - q)p}.
\]

Setting \( u = 1 \) gives \( q = \frac{2p - 1}{2p} \).

7. [25 points] Consider a Markov chain with states \( X_t \in \{1, 2, \ldots, N\} \) for \( t \in \mathbb{N} \), and transition probability matrix with entries:

\[
P_{ij} = \begin{cases} 
  1/(N - 1) & \text{if } i \neq N \text{ and } j \neq i; \\
  1 & \text{if } i = j = N; \\
  0 & \text{otherwise.}
\end{cases}
\]

Let \( T = \min\{t \mid X_t = N\} \). What is \( \mathbb{E}[T \mid X_0 = 1] \)?

Let \( \tau = \mathbb{E}[T \mid X_0 = 1] = \sum_{k=1}^{N} \mathbb{E}[T \mid X_1 = k]P_{1k} = (N - 2)(1 + \tau)/(N - 1) + 1/(N - 1) \),

where the last term is the \( k = N \) term in the sum. Solving for \( \tau \) gives \( N - 1 \). This Markov chain is essentially a “classical version” of Grover’s quantum algorithm. See, for example, M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press 2000) Chapter 6.
8. Suppose the random variables $\xi_i(t) \in \mathbb{N}$ are i.i.d. and have probability generating function

$$\phi(s) = \frac{s + 2}{6 - 3s}.$$ 

Let $X_{t+1} = \xi_1^{(t+1)} + \cdots + \xi_X^{(t+1)}$ for $t \in \mathbb{N}$, with $X_0 = 1$.

a. [10 points] What is $\Pr(X_1 = 0)$?

$$\Pr(X_1 = 0) = \phi(0) = \frac{1}{3}.$$ 

b. [10 points] What is $\mathbb{E}[X_1]$?

$$\mathbb{E}[X_1] = \phi'(1) = \frac{4}{3}.$$ 

c. [10 points] What is $\Pr(X_t > 0$ for all $t \in \mathbb{N})$?

The extinction probability is the non-1 solution to $s = \phi(s)$, i.e., $0 = 3s^2 - 5s + 2 = (3s - 2)(s - 1)$, namely $2/3$. So $\Pr(X_t > 0$ for all $t \in \mathbb{N}) = 1 - 2/3 = 1/3$.

d. [Extra credit: 5 points] What is the probability distribution function for $X_1$?

The probability generating function $\phi(s) = \sum p_k s^k$, so we need to compute the power series expansion of $\phi(s)$:

$$\phi(s) = \frac{s + 2}{6 - 3s} = \frac{s + 2}{6(1 - s/2)} = \frac{s + 2}{6} \left(1 + \frac{s}{2} + \left(\frac{s}{2}\right)^2 + \left(\frac{s}{2}\right)^3 + \cdots\right)$$

$$= \frac{1}{6} \left(2 + 2s + s^2 + \frac{s^3}{2} + \cdots + \frac{s^k}{2^{k-2}} + \cdots\right),$$

so $\Pr(X_1 = 0) = 1/3$ and $\Pr(X_1 = k > 0) = 2^{1-k}/3$. 