1. (10 points) Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^2$ subject to the initial condition $y(0) = 0$. [Hint: Let $z(x) = y(x) + x$.]

This is a first order equation, but it is not separable, linear, or exact. Making the suggested change of variables gives $z' = y' + 1$, so the ODE becomes $z' - 1 = z^2$, or $z' = z^2 + 1$. This equation *is* separable:

$$\int \frac{\mathrm{d}z}{z^2+1} = \int \mathrm{d}x = x + c.$$

To do the first integral, substitute $z = \tan\theta$; $\mathrm{d}z = \sec^2\theta\,\mathrm{d}\theta$ to get:

$$x + c = \int \frac{\sec^2\theta\,\mathrm{d}\theta}{\tan^2\theta + 1} = \int \mathrm{d}\theta = \theta = \arctan z.$$

Taking the tangent of both sides gives $\tan(x+c) = z = y + x$, so $y = \tan(x+c) - x$. Since $y(0) = 0$, $c = 0$, so $y = -x + \tan x$.

2. (15 points) Find the general solution of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \frac{1}{\sin x}$.

This is a second order, linear, constant coefficient inhomogeneous equation. Solving the homogeneous equation first, we write the characteristic equation $r^2+1=0$, which implies $r=\pm i$. Thus $y_h(x) = c_1 \cos x + c_2 \sin x$. To find a particular solution to the inhomogeneous equation, use variation of parameters:

$$\begin{aligned} y &= u(x)\cos x + v(x)\sin x \\ y' &= \underbrace{u'\cos x + v'\sin x}_{=0} - u\sin x + v\cos x \\ y'' &= \underbrace{-u'\sin x + v'\cos x}_{=1/\sin x} - u\cos x - v\sin x. \end{aligned}$$

Solving for u' and v' gives:

$$\begin{aligned} u' &= -1 &\implies\quad u &= -\int \mathrm{d}x = -x + c_1 \\ v' &= \frac{\cos x}{\sin x} &\implies\quad v &= \int \frac{\cos x\,\mathrm{d}x}{\sin x} = \ln|\sin x| + c_2, \end{aligned}$$

so the general solution is $y(x) = c_1\cos x + c_2 \sin x - x\cos x + \ln|\sin x|\sin x$.

3. (15 points) Describe a system in the real world that can be modelled using an ordinary differential equation or a system of ordinary differential equations. Define the variables in the model and write down the equation(s).

Any of the examples from the text, or from lecture, would be an acceptable answer for this question; more interesting answers talked about some of the many other systems that you can model with differential equations. I did require that you described the system, defined the variables, and wrote down the relevant differential equations. I wanted to see an explanation of why each term is in the equation(s). Solving the equations was not necessary, but to the extent that it helped you explain why the differential equation provided a good model of the system, it was worth doing.

4. (5 points) Determine a lower bound for the radius of convergence of a power series solution to $(x^2 - 2x + 2)y'' + y' + xy = 0$ around $x = 0$.

Rewriting this equation so that the coefficient of y'' is 1 gives:

$$y'' + \frac{1}{x^2 - 2x + 2}y' + \frac{x}{x^2 - 2x + 2}y = 0.$$

$x = 0$ is an ordinary point of this ODE since the coefficient functions are defined and continuous there. Each has singularities when $x^2 - 2x + 2 = 0$, *i.e.*, when $x = 1 \pm i$. These points are distance $\sqrt{1^2 + 1^2} = \sqrt{2}$ away from $x = 0$, so a lower bound for the radius of convergence of a power series solution around $x = 0$ is $\sqrt{2}$.

5.a. (3 points) What is the set of ordinary points of the equation $(1+x^2)y''+2xy'-2y=0$? Since $1+x^2 \neq 0$ for all real x, the set of ordinary points is $\mathbb{R}$, all real numbers.

b. (13 points) Find the general solution of the ODE in problem 5.a. This is a second order, linear, homogeneous equation, but the coefficients are not constant. The only systematic solution method we know is by power series. So let

$$y = \sum_{n=0}^{\infty} a_n x^n; \quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}; \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

Plugging these into the ODE gives:

$$\begin{aligned} 0 &= (1+x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2x\sum_{n=0}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n \\ &= \sum_{n=0}^{\infty} \big((n+2)(n+1)a_{n+2} + n(n-1)a_n + 2na_n - 2a_n\big)x^n \end{aligned}$$

$$\implies \quad 0 = (n+2)(n+1)a_{n+2} + \big(n(n-1)+2n-2\big)a_n \quad \implies \quad a_{n+2} = -\frac{n-1}{n+1}a_n$$

Using this recurrence relation we can compute:

$$\begin{aligned} a_2 &= -\frac{-1}{1}a_0 = \frac{1}{1}a_0 \\ a_3 &= 0 \\ \implies \quad a_{2k+1} &= 0 \quad \text{for } 1 \leq k \in \mathbb{Z} \\ a_4 &= -\frac{1}{3}a_2 = -\frac{1}{3}a_0 \\ a_6 &= -\frac{3}{5}a_4 = \frac{1}{5}a_0 \\ \implies \quad a_{2k} &= \frac{(-1)^{k+1}}{2k-1}a_0 \quad \text{for } 1 \leq k \in \mathbb{Z} \end{aligned}$$

So the general solution to the ODE is

$$y(x) = a_1 x + a_0\Big(1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}x^{2k}\Big) = a_1 x + a_0\Big(1 + x\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}x^{2k+1}\Big).$$

You may, although you are not expected to, recognize the series as the Taylor series for $\arctan x$ around $x = 0$. This means that the general solution to the ODE is $y(x) = a_1 x + a_0(1 + x\arctan x)$.

c. (4 points) If you noticed that $y(x) = x$ solves the ODE in problem 5.a, what method (other than the one you used in problem 5.b) could you use to find a second, linearly independent, solution? Plug in the form $y(x) = u(x)x$. This will give a first order equation for $u'(x)$. Alternatively, you could use Abel's theorem to compute the Wronskian and use that to get a first order equation for a second, linearly independent solution.

6.a. (2 points) Compute the Laplace transform of $f(t) = e^{at}$.

$$\int_0^\infty e^{-st}e^{at}\mathrm{d}t = \int_0^\infty e^{(a-s)t}\mathrm{d}t = \frac{e^{(a-s)t}}{a-s}\Big|_0^\infty = \frac{1}{s-a} \quad \text{for } s > a.$$

b. (3 points) Remember that

$$u_c(t) = \begin{cases} 0 & \text{if } t < c; \\ 1 & \text{if } t \geq c. \end{cases}$$

Compute the Laplace transform of $g(t) = u_1(t)e^{a(t-1)}$.

$$\int_0^\infty e^{-st}u_1(t)e^{a(t-1)}\mathrm{d}t = \int_1^\infty e^{-st}e^{a(t-1)}\mathrm{d}t = \int_0^\infty e^{-s(r+1)}e^{ar}\mathrm{d}r \quad \text{where } r = t-1$$

$$= e^{-s}\int_0^\infty e^{-sr}e^{ar}\mathrm{d}r = e^{-s}\mathcal{L}[e^{ar}] = \frac{e^{-s}}{s-a} \quad \text{for } s > a.$$

c. (15 points) Solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y + 1 - 2u_1(t)e^{-(t-1)},$$

when $y(0) = 0$. Taking the Laplace transform of both sides of this equation and setting $Y(s) = \mathcal{L}[y]$ gives:

$$-y(0) + sY(s) = Y(s) + \mathcal{L}[1] - 2\mathcal{L}[u_1(t)e^{-(t-1)}] = Y(s) + \frac{1}{s} - 2\frac{e^{-s}}{s+1} \quad \text{for } s > 0,$$

using the results of parts (a) and (b). Solving for $Y(s)$ gives:

$$Y(s) = \frac{1}{s(s-1)} - 2\frac{e^{-s}}{(s+1)(s-1)} = -\frac{1}{s} + \frac{1}{s-1} + e^{-s}\left(\frac{1}{s+1} - \frac{1}{s-1}\right),$$

where we've used partial fractions twice. Now use the results of parts (a) and (b) again to find the inverse Laplace transforms of the terms on the right hand side and conclude:

$$y(t) = -1 + e^t + u_1(t)(e^{-(t-1)} - e^{t-1}).$$

7. (15 points) Find the general solution to $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$.

First solve the homogeneous equation by finding the eigenvalues and eigenvectors of the coefficient matrix:

$$0 = \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = r^2 - 2r - 3 = (r-3)(r+1) \implies r \in \{3, -1\};$$

$$r = 3 \implies \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0} \implies \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$r = -1 \implies \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0} \implies \mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\implies \mathbf{x}_h = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

Now we could diagonalize the coefficient matrix to solve the inhomogeneous equation, but we can also look for a particular solution of the form $\mathbf{x}_p = \mathbf{u}e^t$. Plugging in gives:

$$\mathbf{u}e^t = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{u}e^t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t \implies \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \implies \mathbf{u} = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix},$$

so the general solution to the equation is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t.$$

8. (Extra credit) Let $F(s) = \mathcal{L}[t^n]$.
   a. (4 points) Compute $F(s)$.

$$\mathcal{L}[t^n] = \int_0^\infty \underbrace{t^n}_{=u} \underbrace{e^{-st}\mathrm{d}t}_{=\mathrm{d}v} = -\frac{t^n}{s}e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\mathrm{d}t.$$

   The first term in the right hand expression vanishes for $s > 0$, so

$$F(s) = \mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}] = \frac{n(n-1)}{s^2}\mathcal{L}[t^{n-1}] = \cdots = \frac{n!}{s^n}\mathcal{L}[1] = \frac{n!}{s^{n+1}} \quad \text{for } s > 0.$$

   b. (1 point) What is $F(1)$? $F(1) = n!/1^{n+1} = n!$.
   c. (5 points) Use the results of problems 8.a and 8.b to create a definition for $(\frac{1}{2})!$. Since $F(1) = n!$, a reasonable definition of $(\frac{1}{2})!$ is $\mathcal{L}[t^{1/2}]$, evaluated at $s = 1$, namely

$$(\frac{1}{2})! = \int_0^\infty e^{-t}t^{1/2}\mathrm{d}t.$$