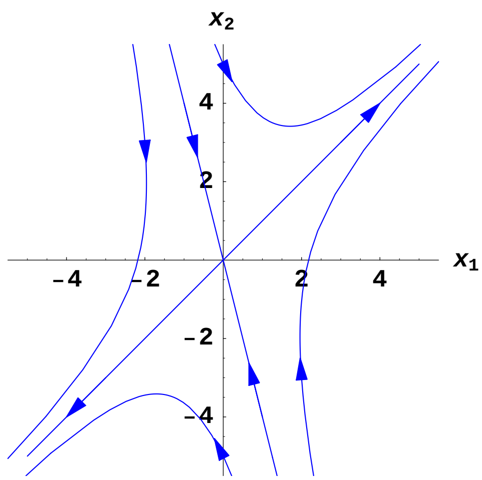
1.a. (20 points) Find the general solution to the system of equations:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 + x_2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 4x_1 - 2x_2.$$

Compute the eigenvalues and eigenvectors of the coefficient matrix:

$$0 = \begin{vmatrix} 1-r & 1 \\ 4 & -2-r \end{vmatrix} = r^2 + r - 6 = (r+3)(r-2) \quad \Longrightarrow \quad r \in \{-3, 2\};$$
$$r = -3 \quad \Longrightarrow \quad \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$
$$r = 2 \quad \Longrightarrow \quad \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\Longrightarrow \quad \mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

b. (10 points) Sketch the solutions to this system of equations in the (x_1, x_2) plane.



c. (10 points) Find the solution that goes through the point $(5, 0)$ at $t = 0$.

$$\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \Longrightarrow \quad \mathbf{x} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} - 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

2. Consider the differential equation

$$(1-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = g(x),$$

where $g(x)$ is an arbitrary function and $0 < x < 1$.

a. (5 points) Show that $y_1(x) = e^x$ and $y_2(x) = x$ solve this equation when $g(x) = 0$.

$$(1-x)e^x + xe^x - e^x = 0 \implies y_1(x) = e^x \text{ solves the equation}$$
$$(1-x)\cdot 0 + x\cdot 1 - x = 0 \implies y_2(x) = x \text{ solves the equation}$$

b. (5 points) Show that the functions $y_1(x)$ and $y_2(x)$ are linearly independent on the interval $0 < x < 1$. Compute the Wronskian:

$$W[y_1, y_2](x) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x(1-x) \neq 0 \text{ for } 0 < x < 1,$$

so these are linearly independent solutions.

c. (20 points) Find the general solution to this equation for an arbitrary function $g(x)$. Hint: Your answer should involve integrals that depend upon $g(x)$, which means that you won't be able to evaluate them since you don't know what $g(x)$ is. Use variation of parameters:

$$y = u(x)e^x + v(x)x; \quad y' = \underbrace{u'e^x + v'x}_{=0} + ue^x + v; \quad y'' = u'e^x + v' + ue^x$$
$$\implies (1-x)(u'e^x + v' + ue^x) + x(ue^x + v) - (u(x)e^x + v(x)x) = g(x)$$
$$\implies u'e^x + v' = g(x)/(1-x)$$

Solving for u' and v' gives:

$$u' = -\frac{xe^{-x}g(x)}{(1-x)^2} \implies u(x) = -\int^x \frac{te^{-t}g(t)}{(1-t)^2}\mathrm{d}t$$
$$v' = \frac{g(x)}{(1-x)^2} \implies v(x) = \int^x \frac{g(t)}{(1-t)^2}\mathrm{d}t$$
$$\implies y(x) = -e^x\int^x \frac{te^{-t}g(t)}{(1-t)^2}\mathrm{d}t + x\int^x \frac{g(t)}{(1-t)^2}\mathrm{d}t$$

3.a. (15 points) Find the general solution to the equation $y''' + 3y'' + 3y' + y = 0$. Try $y(t) = e^{rt}$. Plugging in gives:

$$r^3e^{rt} + 3r^2e^{rt} + 3re^{rt} + e^{rt} = 0 \quad \Longrightarrow \quad 0 = r^3 + 3r^2 + 3r + 1 = (r+1)^3.$$

By analogy with repeated roots for second order equations, this implies that the general solution is $y(t) = (c_1 + c_2t + c_3t^2)e^{-t}$.

b. (15 points) Define a change of variables so that this third order equation is equivalent to a system of 3 first order linear, constant coefficient ODEs, and write down that system of equations.

$$\begin{array}{lcl} x_1 = y & & x_1' = x_2 \\ x_2 = y' & \Longrightarrow & x_2' = x_3 \\ x_3 = y'' & & x_3' = -3x_3 - 3x_2 - x_1 \end{array}$$

4. (Extra credit: 15 points) Find the general solution of the second order differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = \ln x$$

by making the change of variables $x = e^t$ (which leads to a constant coefficient second order linear ODE).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}\right) = e^{-t}\left(e^{-t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - e^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}\right) = e^{-2t}\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

$$\implies e^{2t}e^{-2t}\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right) + 3e^te^{-t}\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = t \quad \implies \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = t$$

This is a constant coefficient inhomogeneous linear equation for $y(t)$, so first solve the homogeneous equation using the characteristic equation:

$$0 = r^2 + 2r - 3 = (r+3)(r-1) \quad \implies \quad y_h(t) = c_1e^{-3t} + c_2e^t.$$

Now try $y_p(t) = at + b$, which implies $t = 2a - 3(at+b) = -3at + (2a - 3b)$, so $a = -1/3$ and $b = -2/9$. Thus

$$y(t) = c_1e^{-3t} + c_2e^t - \frac{1}{3}t - \frac{2}{9}$$

$$\implies y(x) = c_1x^{-3} + c_2x - \frac{1}{3}\ln x - \frac{2}{9}.$$