Name: Section: **21D FINAL**

This is the final from the last time I taught this class. That year the syllabus had more emphasis on sequences and series, covered series expansions around singular points, and did not cover systems of equations. So this is not a perfectly representative sample final. In particular, we have not learned enough to answer the last question in problem 5.a, or to do 5.b. Also, problem 4 is a bad question—it involves too much algebra.

A formula that you might find useful on this test:

$$\big(D - a(t)\big)^{-1}[\cdot] = \frac{1}{\mu(t)} \int^t \mu(s)[\cdot]\mathrm{d}s, \text{ where } \mu(s) = e^{-\int a(s)\mathrm{d}s}.$$

1.a. For each of the following series, explain why it converges or diverges (5 points each).

$$\sum_{n=1}^{\infty} \frac{e^{i\pi n}}{n} \qquad\qquad \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

b. (5 points) What is the radius of convergence of the Taylor series for $\frac{1}{1+x^2}$ around $x = 1$?

2. (15 points) Find the general solution to $y' - 2y = t^2 e^{2t}$.

3.a. (10 points) For the initial value problem $y' = 1 - y^3$, $y(0) = 0$, find the terms up to $t^4$ in a power series solution for $y(t)$.

b. (5 points) Describe another method you could have used to solve this problem.

4.a. (7 points) Solve the initial value problem $y'' + y = \cos(bt)$, $y(0) = 1$, $y'(0) = 0$, for $b \neq 1$.

b. (8 points) Solve the same initial value problem for $b = 1$.

c. (5 points) Plot the solutions to this initial value problem for $b = 0$ and $b = 1$.

5.a. (5 points) For the second order differential equation $xy'' + y' + xy = 0$, which are the ordinary points? The singular points? Are the singular points regular or irregular?

b. (15 points) Find a solution to this equation that satisfies the initial condition $y(0) = 1$.

6.a. (10 points) Suppose $f(t)$ has a Laplace transform $F(s) = \mathcal{L}[f(t)]$, for $s > a \geq 0$. Show that for any constant $c$,

$$\mathcal{L}[e^{ct} f(t)] = F(s-c),$$

for $s > a + c$.

b. (5 points) Find the Laplace transform of the function

$$g(t) = \begin{cases} 0 & \text{for } 0 \leq t < 1; \\ 1 & \text{for } 1 \leq t < 2; \\ 0 & \text{for } 2 \leq t. \end{cases}$$

7. (Extra credit) Consider the homogeneous second order differential equation

$$(x^2 - 1)y'' - 2xy' + 2y = 0.$$

a. (2 points) Check that $y_1(x) = x$ is a solution to this equation.
b. (8 points) Find the general solution to this equation.
c. (10 points) Solve the inhomogenous equation $(x^2 - 1)y'' - 2xy + 2y = (x^2 - 1)^2$.