This is the final from the last time I taught this class. That year the syllabus had more emphasis on sequences and series, covered series expansions around singular points, and did not cover systems of equations. So this is not a perfectly representative sample final. In particular, we have not learned enough to answer the last question in problem 5.a, or to do 5.b. Also, problem 4 is a bad question—it involves too much algebra.

A formula that you might find useful on this test:

$$\big(D - a(t)\big)^{-1}[\cdot] = \frac{1}{\mu(t)} \int^t \mu(s)[\cdot]\mathrm{d}s, \text{ where } \mu(s) = e^{-\int a(s)\mathrm{d}s}.$$

1.a. For each of the following series, explain why it converges or diverges (5 points each).

$$\sum_{n=1}^{\infty} \frac{e^{i\pi n}}{n} \qquad\qquad \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

The first series converges by the alternating series test: $e^{i\pi n} = (-1)^n$ and $\lim 1/n = 0$. The second series also converges, by the comparison test: $e^{1/n} \leq e$ for $n \geq 1$ and $\sum 1/n^p$ converges for $p > 1$ (which you can show using the integral test).

b. (5 points) What is the radius of convergence of the Taylor series for $\frac{1}{1+x^2}$ around $x = 1$? This function is singular at $x = \pm i$, which are distance $\sqrt{2}$ from $x = 1$, so the radius of convergence of the Taylor series around $x = 1$ is $\sqrt{2}$.

2. (15 points) Find the general solution to $y' - 2y = t^2 e^{2t}$. The solution to the homogeneous equation is $y_h(t) = ce^{2t}$, so we can use the method of undetermined coefficients to find a particular solution of the form $y_p(t) = at^3e^{2t}$: $y_p'(t) = a(2t^3e^{2t} + 3t^2e^{2t})$, so plugging into the ODE gives $a(2t^3 + 3t^2)e^{2t} - 2at^3e^{2t} = t^2e^{2t}$. Dividing by t^2e^{2t}, this becomes $a(2t + 3) - 2at = 1$, which implies $a = 1/3$. Thus the general solution is $y(t) = ce^{2t} + \frac{1}{3}t^3e^{2t}$.

3.a. (10 points) For the initial value problem $y' = 1 - y^3$, $y(0) = 0$, find the terms up to t^4 in a power series solution for $y(t)$. Let $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^3 + \cdots$. Plugging this into the ODE gives:

$$\begin{aligned} &a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots \\ &\quad = 1 - (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^3 + \cdots)^3 \\ &\quad = 1 - \left(a_0^3 + 3a_0^2 a_1 t + (3a_0^2 a_2 + 3a_0 a_1^2)t^2 + (3a_0^2 a_3 + 6a_0 a_1 a_2 + a_1^3)t^3 + \cdots\right). \end{aligned}$$

Equating coefficients of t^n on both sides of the equation, and using the initial condition $y(0) = a_0 = 0$ gives:

$$\begin{aligned} a_1 &= 1 - a_0^3 \\ 2a_2 &= -3a_0^2 a_1 \\ 3a_3 &= -(3a_0^2 a_2 + 3a_0 a_1^2) \\ 4a_4 &= -(3a_0^2 a_3 + 6a_0 a_1 a_2 + a_1^3) \end{aligned} \implies \begin{aligned} a_1 &= 1 \\ a_2 &= 0 \\ a_3 &= 0 \\ a_4 &= -a_1/4 = -1/4 \end{aligned}$$

Thus up to the t^4 term, $y(t) = t - t^4/4 + \cdots$.

b. (5 points) Describe another method you could have used to solve this problem. The equation is separable, so

$$\int \frac{\mathrm{d}y}{1 - y^3} = \int \mathrm{d}t.$$

The y integral is a little difficult to compute, but eventually gives:

$$\frac{1}{\sqrt{3}} \arctan\left(\frac{2y+1}{\sqrt{3}}\right) - \frac{1}{3}\ln|y-1| + \frac{1}{6}\ln|y^2 + y + 1| = t + c.$$

4.a. (7 points) Solve the initial value problem $y'' + y = \cos(bt)$, $y(0) = 1$, $y'(0) = 0$, for $b \neq 1$. The characteristic equation for the homogeneous equation is $r^2 + 1 = 0$, so $r = \pm i$. Thus $y_h(t) = c_1 \cos t + c_2 \sin t$. When $b \neq 1$, the inhomogeneous term $\cos(bt)$ is not part of the solution to the homogeneous equation, so a particular solution to the inhomogeneous equation has the form: $y_p(t) = A\cos(bt) + B\sin(bt)$; $y_p'(t) = -Ab\sin(bt) + Bb\cos(bt)$; $y_p''(t) = -Ab^2\cos(bt) - Bb^2\sin(bt)$. Plugging these into the inhomogenous equation gives:

$$-Ab^2\cos(bt) - Bb^2\sin(bt) + A\cos(bt) + B\sin(bt) = \cos(bt).$$

Since this must hold for all t, it must in particular hold for $t = 0$ and $t = \pi/(2b)$. At these values we get

$$-Ab^2 + A = 1 \quad \Longrightarrow \quad A = \frac{1}{1-b^2};$$
$$-Bb^2 + B = 0 \quad \Longrightarrow \quad B = 0.$$

So the general solution to the inhomogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{1-b^2}\cos(bt).$$

Then the initial conditions imply that

$$1 = y(0) = c_1 + \frac{1}{1-b^2} \quad \Longrightarrow \quad c_1 = \frac{-b^2}{1-b^2}$$

and $0 = y'(0) = c_2$. So finally, the solution to the inhomogeneous equation that satisfies the initial condition is:

$$y(t) = \frac{-b^2}{1-b^2}\cos t + \frac{1}{1-b^2}\cos(bt).$$

b. (8 points) Solve the same initial value problem for $b = 1$. Now the inhomogeneous term is part of the solution to the homogeneous equation, so a particular solution to the inhomogeneous equation has the form: $y_p(t) = t(A\cos t + B\sin t)$; $y_p'(t) = t(-A\sin t + B\cos t) + (A\cos t + B\sin t)$; $y_p''(t) = t(-A\cos t - B\sin t) + 2(-A\sin t + B\cos t)$. Plugging these into the inhomogenous equation gives:

$$t(-A\cos t - B\sin t) + 2(-A\sin t + B\cos t) + t(A\cos t + B\sin t) = \cos t,$$

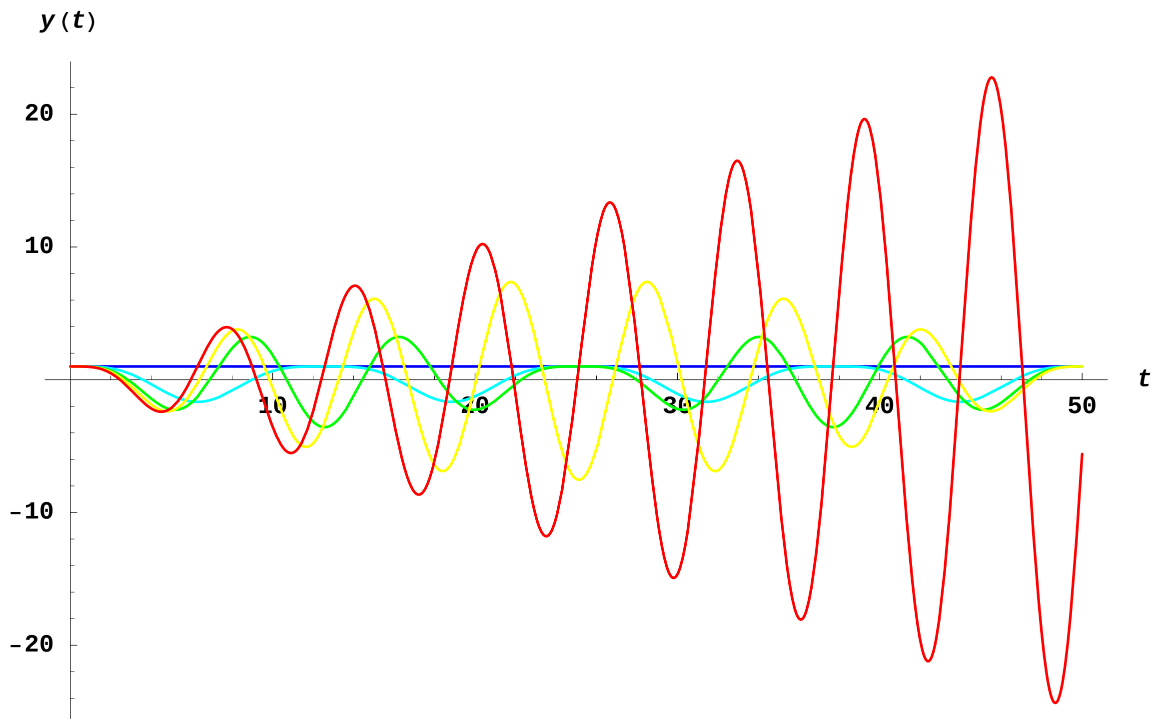
which simplifies to $2(-A\sin t + B\cos t) = \cos t$. Thus $A = 0$ and $B = 1/2$, so the general solution to the inhomogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{2} t \sin t.$$

Then the initial conditions imply that $1 = y(0) = c_1$ and $0 = y'(0) = c_2$. So finally, the solution to the inhomogeneous equation that satisfies the initial condition is:

$$y(t) = \cos t + \frac{1}{2} t \sin t.$$

c. (5 points) Plot the solutions to this initial value problem for $b = 0$ and $b = 1$. In the following plot $b = 0$ is blue, $b = 1/2$ is teal, $b = 3/4$ is green, $b = 7/8$ is yellow, and $b = 1$ is red.



5.a. (5 points) For the second order differential equation $xy'' + y' + xy = 0$, which are the ordinary points? The singular points? Are the singular points regular or irregular? The set of ordinary points is $\{x \in \mathbb{R} \mid x \neq 0\}$. $x = 0$ is the only singular point. We have not learned the definition of regular or irregular singular points.

b. (15 points) Find a solution to this equation that satisfies the initial condition $y(0) = 1$. Since $x = 0$ is a singular point, we have not learned how to find a power series solution around it.

6.a. (10 points) Suppose $f(t)$ has a Laplace transform $F(s) = \mathcal{L}[f(t)]$, for $s > a \geq 0$. Show that for any constant c,

$$\mathcal{L}[e^{ct}f(t)] = F(s-c),$$

for $s > a + c$. From the definition:

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{-st}e^{ct}f(t)\mathrm{d}t = \int_0^\infty e^{-(s-c)t}f(t)\mathrm{d}t = F(s-c),$$

provided the integral converges. By hypothesis, it does so for $s - c > a$, which is equivalent to $s > a + c$.

b. (5 points) Find the Laplace transform of the function

$$g(t) = \begin{cases} 0 & \text{for } 0 \leq t < 1; \\ 1 & \text{for } 1 \leq t < 2; \\ 0 & \text{for } 2 \leq t. \end{cases}$$

From the definition

$$\mathcal{L}[g(t)] = \int_0^\infty e^{-st}g(t)\mathrm{d}t = \int_1^2 e^{-st}\mathrm{d}t = \left.\frac{-e^{-st}}{s}\right|_1^2 = (e^{-s} - e^{-2s})\frac{1}{s}.$$

7. (Extra credit) Consider the homogeneous second order differential equation

$$(x^2-1)y'' - 2xy' + 2y = 0.$$

a. (2 points) Check that $y_1(x) = x$ is a solution to this equation.

$$(x^2-1)\cdot 0 - 2x\cdot 1 + 2x = 0 \quad \Longrightarrow \quad y_1(x) = x \text{ solves the equation.}$$

b. (8 points) Find the general solution to this equation. Try $y(x) = u(x)x$; $y'(x) = u'(x)x + u(x)$; $y''(x) = u''(x)x + 2u'(x)$. Plugging into the ODE gives

$$0 = (x^2-1)(u''x+2u') - 2x(u'x+u) + 2ux = x(x^2-1)u'' - 2u'.$$

Let $v = u'$. Then $x(x^2-1)v' = 2v \quad \Longrightarrow$

$$\int \frac{\mathrm{d}v}{v} = \int \frac{2\mathrm{d}x}{x(x^2-1)} \quad \Longrightarrow \quad \ln|v| = \ln\left|\frac{x^2-1}{x^2}\right| + c \quad \Longrightarrow \quad u' = C(1-x^{-2})$$

Integrating once more gives $u(x) = C(x+x^{-1}) + D$, so $y(x) = C(x^2+1) + Dx$.

c. (10 points) Solve the inhomogenous equation $(x^2-1)y'' - 2xy + 2y = (x^2-1)^2$. Use variation of parameters:

$$y = C(x)(x^2+1) + D(x)x; \quad y' = \underbrace{C'(x^2+1) + D'x}_{=0} + 2xC + D; \quad y'' = \underbrace{2xC' + D'}_{=x^2-1} + 2C$$

Solving for C' and D' gives

$$\begin{aligned} C' = x \quad &\Longrightarrow \quad C(x) = \frac{1}{2}x^2 + c_1 \\ D' = -(x^2+1) \quad &\Longrightarrow \quad D(x) = -\frac{1}{3}x^3 - x + c_2 \\ \Longrightarrow \quad y(x) &= \left(\frac{1}{2}x^2 + c_1\right)(x^2+1) + \left(-\frac{1}{3}x^3 - x + c_2\right)x \\ &= c_1(x^2+1) + c_2 x + \frac{1}{6}x^4 - \frac{1}{2}x^2 \end{aligned}$$