21D (NOT EXACTLY SAMPLE) EXAM PROBLEM ANSWERS

2.a. The characteristic equation for the homogeneous equation is $r^2 + 1 = 0$, so $r = \pm i$. Thus the general solution to the homogeneous equation is:

$$y(t) = c_1 \cos t + c_2 \sin t.$$

When $b \neq 1$, the inhomogeneous term $\cos(bt)$ is not part of the solution to the homogeneous equation, so a particular solution to the inhomogeneous equation has the form:

$$\begin{aligned} y &= A\cos(bt) + B\sin(bt) \\ \Rightarrow y' &= -Ab\sin(bt) + Bb\cos(bt) \\ \Rightarrow y'' &= -Ab^2\cos(bt) - Bb^2\sin(bt). \end{aligned}$$

Plugging these into the inhomogenous equation gives:

$$-Ab^2\cos(bt) - Bb^2\sin(bt) + A\cos(bt) + B\sin(bt) = \cos(bt).$$

Since this must hold for all t, it must in particular hold for $t = 0$ and $t = \pi/(2b)$. At these values we get

$$\begin{aligned} -Ab^2 + A = 1 &\Rightarrow A = \frac{1}{1-b^2}; \\ -Bb^2 + B = 0 &\Rightarrow B = 0. \end{aligned}$$

So the general solution to the inhomogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{1-b^2}\cos(bt).$$

Then the initial conditions imply that

$$1 = y(0) = c_1 + \frac{1}{1-b^2} \quad \Rightarrow \quad c_1 = \frac{-b^2}{1-b^2}$$

and $0 = y'(0) = c_2$. So finally, the solution to the inhomogeneous equation that satisfies the initial condition is:

$$y(t) = \frac{-b^2}{1-b^2}\cos t + \frac{1}{1-b^2}\cos(bt).$$

b. When $b = 1$, the inhomogeneous term is part of the solution to the homogeneous equation, so a particular solution to the inhomogeneous equation has the form:

$$\begin{aligned} y &= t(A\cos t + B\sin t) \\ \Rightarrow y' &= t(-A\sin t + B\cos t) + (A\cos t + B\sin t) \\ \Rightarrow y'' &= t(-A\cos t - B\sin t) + 2(-A\sin t + B\cos t). \end{aligned}$$

Plugging these into the inhomogenous equation gives:

$$t(-A\cos t - B\sin t) + 2(-A\sin t + B\cos t) + t(A\cos t + B\sin t) = \cos t,$$

which simplifies to $2(-A\sin t + B\cos t) = \cos t$. Thus $A = 0$ and $B = 1/2$, so the general solution to the inhomogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{2} t \sin t.$$

Then the initial conditions imply that $1 = y(0) = c_1$ and $0 = y'(0) = c_2$. So finally, the solution to the inhomogeneous equation that satisfies the initial condition is:

$$y(t) = \cos t + \frac{1}{2} t \sin t.$$

c. In the following plot $b = 1/2$ is blue, $b = 3/4$ is green, $b = 7/8$ is yellow, and $b = 1$ is red.