Math 31a Extra Credit 3 (due Monday, 16 November 2009)

Let $\mathrm{O}(n) = \{A \in \mathrm{Mat}(n,n) \mid \forall \vec{v}, \vec{w} \in \mathbb{R}^n, (A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}\}$. That is, $\mathrm{O}(n)$ consists of all the $n \times n$ real matrices that leave the dot product invariant.

1. Show that $\mathrm{O}(n) = \{A \in \mathrm{Mat}(n,n) \mid A \text{ is orthogonal}\}$.

2. Show that $\mathrm{O}(n)$ is a *group under matrix multiplication*, *i.e.*:

   a. $I_n \in \mathrm{O}(n)$;

   b. if $A \in \mathrm{O}(n)$ then A^{-1} exists and is in $\mathrm{O}(n)$;

   c. $\mathrm{O}(n)$ is closed under matrix multiplication, *i.e.*, if $A, B \in \mathrm{O}(n)$ then $AB \in \mathrm{O}(n)$.

3. Find an explicit description of the matrices in $\mathrm{O}(2)$.

4. Let $\mathrm{SO}(3) = \{A \in \mathrm{O}(3) \mid \forall \vec{v}, \vec{w} \in \mathbb{R}^3, (A\vec{v}) \times (A\vec{w}) = A(\vec{v} \times \vec{w})\}$. That is, $\mathrm{SO}(3)$ consists of 3×3 real matrices that leave the dot product invariant and for which the cross product is *covariant*.

   a. Show that $\mathrm{SO}(3)$ is a group.

   b. Show that $\mathrm{SO}(3) = \{A \in \mathrm{O}(3) \mid \det A = 1\}$.

5. Give a geometrical interpretation of $\mathrm{O}(3)$ and $\mathrm{SO}(3)$.