Problem Set #1

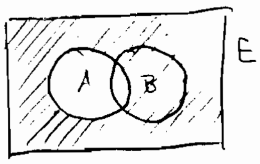
Section 0.3

(0.3.1) Let E be a set, with subsets $A \subset E$ and $B \subset E$, and let $*$ be the operation $A * B = (E-A) \cap (E-B)$. Express the sets below using only A, B, and $*$.

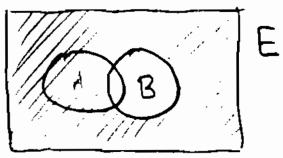
(a) $A \cup B$

I will do part (a) in somewhat greater detail than parts (b) and (c). The first thing I want to do is to draw a few pictures to get an idea what $A * B$ looks like. Note that pictures ARE NOT A PROOF! Pictures are an excellent way to start a problem and get some intuition, but we will need to back them up.

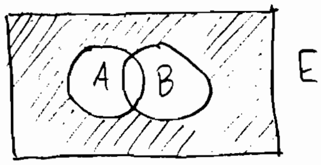
$E-A$



$E-B$



$A * B = (E-A) \cap (E-B)$



So it looks like when we $*$ two sets, it takes their union and then takes the complement of that union. Thus, to get $A \cup B$, we need to apply $*$ to two sets whose union is the complement of $A \cup B$. But in the last picture above, $A * B$ is the complement of $A \cup B$, so my guess is that

$$A \cup B = (A * B) * (A * B)$$

Proof: I will use the notation A^c instead of $E-A$ to denote a complement. It will make the proof easier to read:

$$
\begin{aligned}
(A * B) * (A * B) &= (A * B)^c \cap (A * B)^c && \text{(apply definition of } *) \\
&= (A * B)^c && (S \cap S = S \text{ for any set } S) \\
&= (A^c \cap B^c)^c && \text{(apply definition of } * \text{ again)} \\
&= ((A \cup B)^c)^c && \text{(De Morgan's Law)} \\
&= A \cup B && ((S^c)^c = S \text{ for any set } S)
\end{aligned}
$$

Thus $(A * B) * (A * B) = A \cup B$, so the proof is done.

(0.3.1 continued)

(b) $A \cap B$

For parts (b) and (c), I will skip drawing the pictures. This step is how you get your guess. I'll jump right to claiming the answer and then proving it.

Claim: $A \cap B = (A * A) * (B * B)$

Proof: Note that $A * A = A^c \cap A^c = A^c$, so similarly $B * B = B^c \cap B^c = B^c$. Thus

$$\begin{aligned} (A * A) * (B * B) &= (A^c) * (B^c) && \text{(by the above)} \\ &= (A^c)^c \cap (B^c)^c && \text{(by definition of } *) \\ &= A \cap B && \text{(since } (S^c)^c = S \text{ for any set } S) \end{aligned}$$

Done.

(c) $E - A$ (equivalently A^c)

We actually did this one by accident in the course of part (b).

$$\begin{aligned} A * A &= (E - A) \cap (E - A) && \text{(by definition of } *) \\ &= E - A && \text{(since } S \cap S = S \text{ for any set } S) \end{aligned}$$

Done.

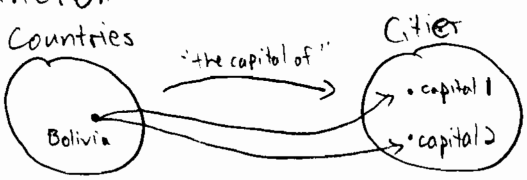
Section 0.4

(0.4.1): Are the following true functions? That is, are they both everywhere defined and well defined?

(a) "the aunt of," from people to people
This relation is not true function. It sends a person to his or her aunt. If person A does not have an aunt, then the relation does not send A anywhere, so it is not everywhere defined. Also, if person B has multiple aunts, then to whom shall we send B? On person B, this relation is not well-defined.

(b) $f(x) = \frac{1}{x}$, from the real numbers to the real numbers
This relation is also not a function, since $f(0)$ is not defined.

(c) "The capital of", from countries to cities (careful - at least two countries, the Netherlands and Bolivia, have two capitals)
This relation would send Bolivia to two cities, so it is not well-defined, hence not a true function



Not a function.

(0.4.2): Of the relations in exercise 0.4.1, which are onto? One-to-one?

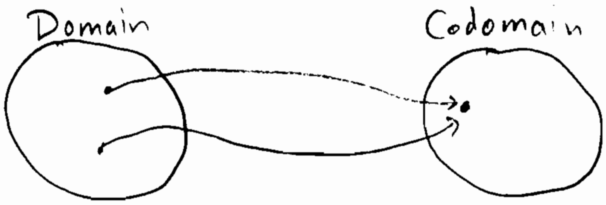
Onto: A function (or relation) is onto if each element in the codomain is mapped to by atleast one element of the domain.

- For part (a), I am a person, so I am an element of the codomain. However, being a guy, I am not the aunt of anyone. Thus, no people in the domain map to me, so the relation is not onto
- For part (b), there is no real number x such that $\frac{1}{x} = 0$. Thus, 0 in the codomain is not mapped to, so the relation is not onto.
- For part (c), not every city is the capital of a country. San Diego is in the codomain, but it is not the capital of a country, so no country maps to San Diego. Hence, not onto

(continues on next page) ⟶

(0.4.2 continued)

One-to-one: A function or relation is one-to-one if each element in the codomain is mapped to by at most one element of the domain. Said another way, it is one-to-one if there are no elements of the codomain which are mapped to by multiple elements of the domain.



(not one-to-one)

- For part (a), suppose that aunt P has multiple nieces and nephews. Then each of these nieces and nephews will be sent to P in the codomain. Hence the relation is not one-to-one.
- For part (b), it possible that there is a $y \in \mathbb{R}$ (the codomain) such that multiple elements of the domain map to y? Suppose so, that is, suppose $y = f(a)$ and $y = f(b)$ for some $a, b \in \mathbb{R}$ (the domain). Then

$$y = \frac{1}{a} \quad \text{and} \quad y = \frac{1}{b}$$

Thus $\frac{1}{a} = \frac{1}{b}$, so $a = b$. Hence only one element can map to any element of the codomain, so $f(x) = 1/x$ is one-to-one.

- For part (c), no city is the capital of more than one country (I think), so no city in the codomain will be mapped to by multiple countries in the domain. Hence the relation is one-to-one.

(0.4.3) (a) Make up a nonmathematical transformation which is bijective (onto and one-to-one)

There are lots of answers, but I will give just one example.

Let A be the set with only one element, namely me. So $A = \{me\}$.

Let B be the set with only one kind of food, a turkey sandwich. So $B = \{\text{turkey sandwich}\}$.

Let $f: A \to B$ be "what I had for lunch. Then f is a function from A to B.

I did indeed have a turkey sandwich for lunch, so $f(me) = $ turkey sandwich.

Then every element in the codomain (there's only one, the sandwich) is mapped to by exactly one element of the domain (namely me). Hence f is onto and one-to-one.

(b) Make up a mathematical transformation that is bijective.

Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$. The every element of the codomain is mapped to by exactly one element of the domain, namely itself. Hence f is one-to-one and onto.

(0.4.4)

(a) Make up a non-mathematical transformation which is onto, but not one-to-one.

Let A be the set of all people, so $A = \{\text{people}\}$, and B be the set with seven elements, the days of the week. Let $f: A \to B$ be the transformation sending each person to the day of the week they were born on. Then f is onto because every element of the codomain lies in the image of f, that is, for each day of the week, there is someone born on that day. But f is not one-to-one, because there is more than one person on Earth that was born on a Monday, for example.

(b) Make up a mathematical transformation which is onto, but not one-to-one.

Let $f(x) = x^2$ be a function from the real numbers to the non-negative real numbers. That is, the domain is $\mathbb{R}$, and the codomain is all numbers greater than or equal to zero. Then f is onto, because every non-negative number has a square root, but it is not one-to-one, because $f(-2) = f(2) = 4$, for example, so multiple distinct numbers may map to the same element of the codomain. (Another way of saying this is that f fails the horizontal line test).

(0.4.6): The transformation $f(x)=x^2$ from real numbers to positive real numbers is onto but not one-to-one. (Note that they do not include zero in the codomain. I'm not sure if this was on accident, but we'll take them at their word).

(a) Can you make it one-to-one by changing its domain? By changing its codomain?

Every single number n in the codomain is mapped to by exactly two real numbers in the domain, namely $\pm\sqrt{n}$. Thus if we restrict the domain to be only positive real numbers (or alternatively only negative real numbers), then f will be one-to-one.

However, the given function sends two real numbers to every positive real number, so there is no way to change the codomain (by throwing some things out) to make f one-to-one.

(b) Can you make it not onto by changing its domain? By changing its codomain?

If we remove both m and $-m$ from the domain for any $m \in \mathbb{R}$, then nothing will map to m^2 in the codomain, so f will not be onto.

There is no way to restrict the codomain to make f not onto. Every positive real number is mapped to, so if we throw out some elements of the codomain, it will remain onto. I suppose we could do something silly like add my left shoe to the codomain. Then nothing would map to this element, but I doubt this is what the book had in mind.