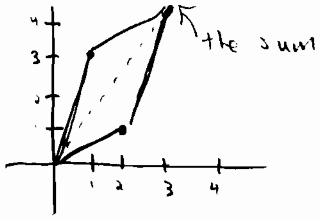
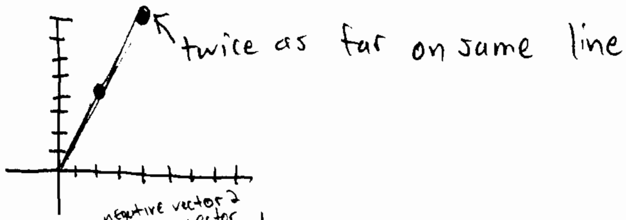
Section 1.1

(1.1.1) Compute the following vectors by coordinates and sketch what you did.

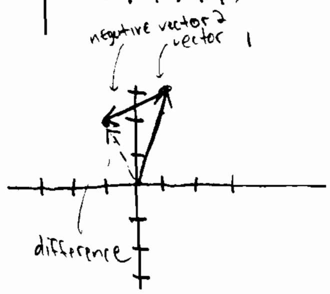
(a) $\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$



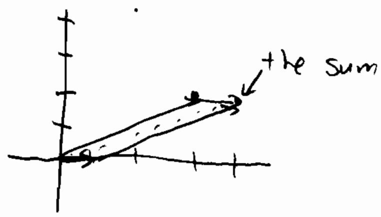
(b) $2\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}4\\8\end{bmatrix}$



(c) $\begin{bmatrix}1\\3\end{bmatrix}-\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$



(d) $\begin{bmatrix}3\\2\end{bmatrix}+\vec{e}_1=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$

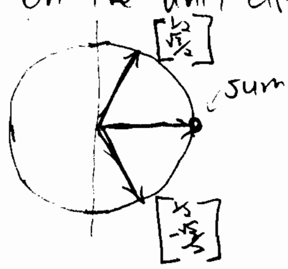


(1.1.4) (a) Name the two trivial subspaces of $\mathbb{R}^n$.

The two trivial subspaces are the zero vector space $\{\vec{0}\}$ and $\mathbb{R}^n$ itself.

(b) Let $S^1\subset\mathbb{R}^2$ be the unit circle with equation $x^2+y^2=1$. Do there exist any two elements of S^1 whose sum is an element of S^1?

Yes there are such pairs. Many, in fact, though I will only give one such pair. Note that $\begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix}$ and $\begin{bmatrix}1/2\\ -\sqrt{3}/2\end{bmatrix}$ are in S^1, corresponding to the angles $\pi/3$ and $-\pi/3$ respectively. Their sum is $\begin{bmatrix}1\\0\end{bmatrix}$, also on the unit circle. See picture below.



(1.1.5) Using sum notation (discussed in section 0.1) write the vectors in the left margin as a sum of multiples of standard basis vectors.

(a) $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ We don't actually know how many terms this vector has, but I will assume that it has n terms. Then

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \vec{e}_1 + \vec{e}_2 + \cdots + \vec{e}_n = \sum_{k=1}^{n} \vec{e}_k$$

(b) $$\begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + n\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \vec{e}_1 + 2\vec{e}_2 + \cdots + n\vec{e}_n = \sum_{k=1}^{n} k\vec{e}_k$$

(c) $\begin{bmatrix} 3 \\ 4 \\ \vdots \\ n \end{bmatrix}$ Note that this vector only has $n-2$ terms in it. Then

this is $\vec{e}_{n-2}$

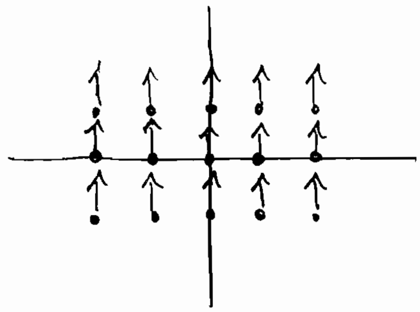
$$\begin{bmatrix} 3 \\ 4 \\ \vdots \\ n \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + n\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= 3\vec{e}_1 + 4\vec{e}_2 + \cdots + n\vec{e}_{n-2}$$

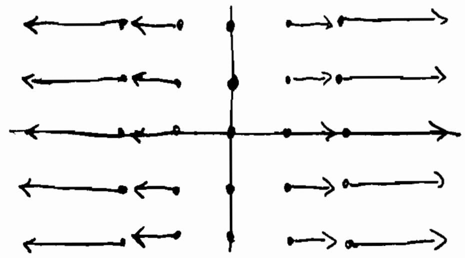
$$= \sum_{k=1}^{n-2} (k+2)\vec{e}_k$$

(1.1.6) Sketch the following vector fields.

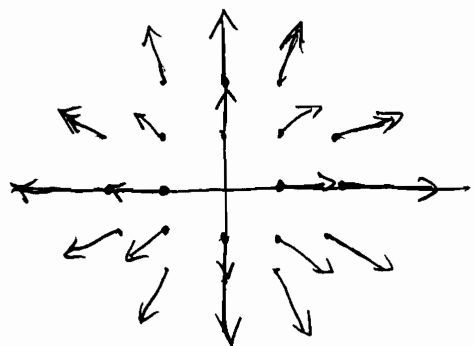
(a) $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



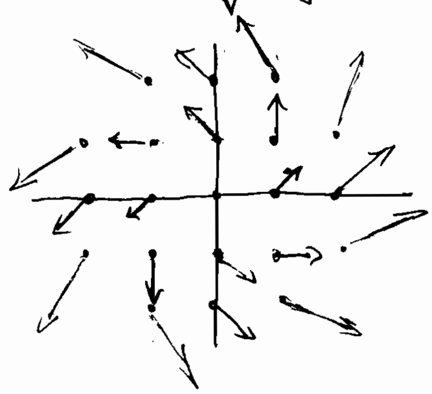
(b) $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$



(c) $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$



(h) $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$



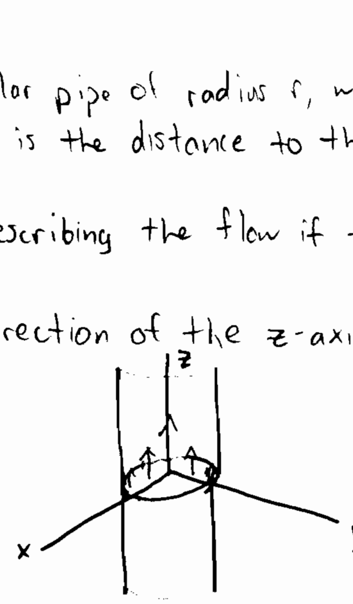
I've shortened the length of the arrows to make it more readable.

A spiral outward.

(1.1.8) Suppose that in a circular pipe of radius r, water is flowing in the direction of the pipe with speed r^2-a^2, where a is the distance to the axis of the pipe.

(a) Write the vector field describing the flow if the pipe is in the direction of the z-axis.

If the pipe lies in the direction of the z-axis, then it looks something like



Now the water is either flowing up or down. The problem didn't tell us which, so I will assume that water flows up, in the direction of the positive z-axis. Intuitively, the water flows fastest in the center of the pipe where r^2-a^2 is largest because $a=0$. Along the walls of the pipe, the water slows to a stop.

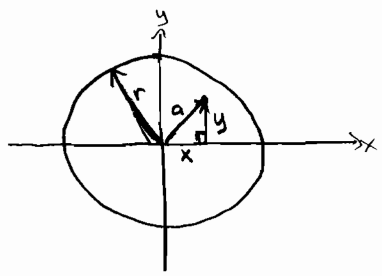
Our velocity vector field has the form

$$\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

where the three components give the flow in the x, y, and z directions respectively. Since no water flows in the x or y directions (only up), those components are zero. Thus the entire velocity lies in the z-component, so

$$\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ r^2-a^2 \end{bmatrix}$$

We're not quite done yet. The r is a constant, so that's fine, but a is a variable that depends on x, y, and z, so we must replace it. Specifically, since a is the distance to the z-axis,



$$a^2 = x^2+y^2$$

by the Pythagorean Theorem. Substituting this in, we get that the velocity vector field is given by

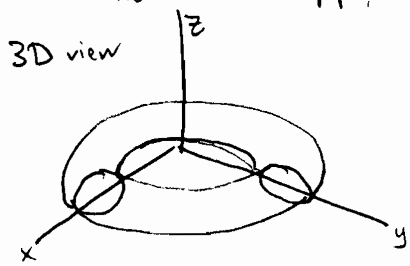
$$\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ r^2-(x^2+y^2) \end{bmatrix}$$

If we had chosen the water to flow downword, we would have gotten the negative of my answer. Either result is fine.

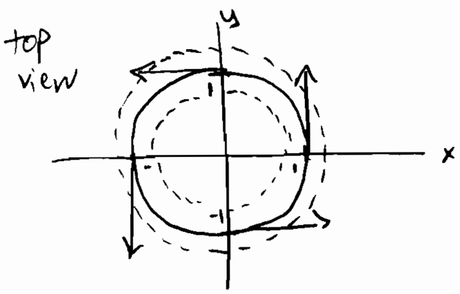
(1.1.8 continued)

(b) Write the vector field describing the flow if the axis of the pipe is the unit circle in the xy-plane.

If the axis of the pipe is the unit circle in the xy-plane, with the axis running down the center of the pipe, then this pipe is shaped like a doughnut (or a torus).



Now the water flows in the direction of the pipe, which could either be clockwise or counterclockwise as seen from above. I will assume that the flow is counterclockwise.



We know that the flow is around in a circle, and not at all in the z-direction, so the velocity vector field is of the form

$$\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} ? \\ ? \\ 0 \end{bmatrix}$$

Now we know that the full vector field has magnitude r^2-a^2, so I will write $\vec{F}$ as

$$\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (r^2-a^2)\begin{bmatrix} ? \\ ? \\ 0 \end{bmatrix}$$

where the bracketed piece should have magnitude (length) equal to 1. Let's figure out the bracketed piece first. We need a vector field that will give counter-clockwise rotation. I claim that $\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$ will work

You have actually discussed this vector field in class, but alternatively, you could arrive at this by using trig, dot products, or the fact that in two dimensions, the slopes of two perpendicular lines are negative reciprocals of each other. I discussed each of these in section and office hours, so I'll leave out that explanation here.

Now we have a slight problem: $\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}$ does not have magnitude of one. In fact, it's length is $\sqrt{x^2+y^2}$, the standard distance formula applied to our vector. Thus, in order to get length one, we must normalize (rescale) our vector by dividing by the length. Hence

$$\begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \\ 0 \end{bmatrix}$$

is the bracketed part we want.

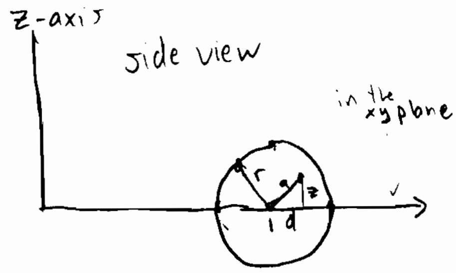
(continued) ⟶

(1.1.8 part (b), continued.

Thus we have for our velocity vector field

$$\vec{F}\begin{pmatrix}x\\y\\z\end{pmatrix} = (r^2-a^2)\begin{bmatrix}\frac{-y}{\sqrt{x^2+y^2}}\\ \frac{x}{\sqrt{x^2+y^2}}\\ 0\end{bmatrix}$$

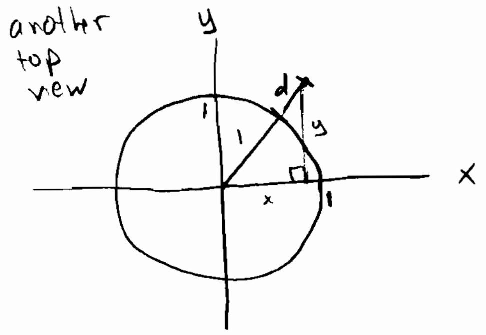
Again, we should get rid of the a^2, since a can be written in terms of x, y, z.



The picture to the left shows a cross-section of our doughnut. We know that

$$a^2 = z^2 + d^2$$

where d is marked in the picture as the horizontal component of a.



Then again by the Pythagorean Theorem,

$$(1+d)^2 = x^2+y^2,$$

So

$$1+d = \sqrt{x^2+y^2}$$
$$d = -1+\sqrt{x^2+y^2}$$
$$d^2 = 1+x^2+y^2-2\sqrt{x^2+y^2}$$

Substituting this back in to the formula $a^2=z^2+d^2$, we get

$$a^2 = 1+x^2+y^2+z^2-2\sqrt{x^2+y^2}$$

Thus, finally, we get that the velocity vector field is given by

$$\vec{F}\begin{pmatrix}x\\y\\z\end{pmatrix} = \left(r^2-\left(1+x^2+y^2+z^2-2\sqrt{x^2+y^2}\right)\right)\begin{bmatrix}\frac{-y}{\sqrt{x^2+y^2}}\\ \frac{x}{\sqrt{x^2+y^2}}\\ 0\end{bmatrix}$$

You can multiply it all out if you wish. If you had taken the flow of water to be clockwise instead, the result would be the negative of mine. Either answer is fine.

Section 1.2

(1.2.2) Perform the following matrix multiplications when possible.

(a) $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 9 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 28 & 14 \\ 79 & 44 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & 3 \\ -2 & 2 \end{bmatrix}$$ Not possible; matrix dimensions are incompatible.

(c) $$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 2 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -5 \\ 4 & -1 & -3 \\ 1 & 0 & 1 \end{bmatrix}$$

(d) $$\begin{bmatrix} 7 & 1 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 31 \\ -5 \\ -2 \end{bmatrix}$$

(e) $$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 10 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -10 & 29 \\ -9 & 24 \end{bmatrix}$$ I multiplied the first two matrices first, then this result agoint the last.

(f) $$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix}$$ Not possible; matrix dimensions are incompatible.

I only just now saw the note on the side of the problem to do the matrix multiplication in the format

$$\begin{array}{cc} & \begin{bmatrix} B \end{bmatrix} \\ \begin{bmatrix} A \end{bmatrix} & \begin{bmatrix} A\cdot B \end{bmatrix} \end{array}$$

Sorry for not using that formatting convention.

(1.2.3) Given the matrices in the margin at left,

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

(a) Compute the third column of AB without computing the entire matrix AB.

Following the explanation on page 45, the third column of AB is equal to A times the third column of B:

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

So the third column of AB is equal to $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

(b) Compute the second row of AB without computing the entire matrix AB.

Again, by the reasoning on page 45, the second row of AB is equal to the second row of A times B:

$$\begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 16 & 2 \end{bmatrix}$$

(1.2.5) Let A and B be $n \times n$ matrices, with A symmetric. Are the following true or false?

We will need to justify our answers. Note that since A is symmetric, $A^T = A$.

(a) $(AB)^T = B^T A$?

$(AB)^T = B^T A^T = B^T A$ so true.

(b) $(A^T B)^T = B^T A^T$?

$(A^T B)^T = B^T (A^T)^T = B^T A = B^T A^T$ so true.

(1.2.5 continued)

(c) $(A^TB)^T = BA$?

$$(A^TB)^T = B^T(A^T)^T = B^TA \neq BA$$

This seems to be false. As a specific example, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then $B^TA = B^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, but $BA = B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and these are not equal.

(d) $(AB)^T = A^TB^T$?

$$(AB)^T = B^TA^T \neq A^TB^T$$

This again looks like the claim is false. For a specific counterexample, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Then $B^TA^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

But $A^TB^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$. These are not equal, so the claim is indeed false.

(1.2.8) For what values of a do the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

satisfy $AB = BA$?

We just calculate AB and BA to find what a must be.

AB:
$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1+a & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

BA:
$$\begin{array}{cc} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ a+1 & a \end{bmatrix} \end{array}$$

So $\begin{bmatrix} 1+a & 1 \\ 1 & 0 \end{bmatrix}$ must equal $\begin{bmatrix} 1 & 1 \\ a+1 & a \end{bmatrix}$. The only way this can happen is if $a=0$.

(1.2.10) What is the inverse of the matrix $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ for $a \neq 0$?

We are given a formula for the inverse of a 2×2 matrix on page 48. Specifically, if $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, then $B^{-1} = \dfrac{1}{xw-yz} \begin{bmatrix} w & -y \\ -z & x \end{bmatrix}$ as long as $xw - yz \neq 0$.

Using this formula, $A^{-1} = \dfrac{1}{a^2} \begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}$, and since $a \neq 0$, the formula makes sense.

(1.2.15) Show that
$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ has an inverse of the form $B = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$. Find it.

I will calculate AB to solve for the unknowns in B to get $AB = $ Identity.

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & y+az+b \\ 0 & 1 & c+z \\ 0 & 0 & 1 \end{bmatrix}$$

So forcing $AB = $ Identity, we get
$$a + x = 0$$
$$y + az + b = 0$$
$$c + z = 0$$

This implies that
$$x = -a$$
$$y = ac - b$$
$$z = -c$$
will make $AB = Id$.

To show that these choices of x, y, z make $B = A^{-1}$, we also must show that $BA = Id$.

$$\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So $A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$.

(1.2.16) Show that if A is any matrix, then $A^T A$ is symmetric.

First of all, if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so $A^T A$ at least makes sense (it is a product we can multiply). Now showing that $A^T A$ is symmetric amounts to showing that

$$(A^T A)^T = A^T A.$$

Let's try.

$$\begin{aligned}(A^T A)^T &= A^T (A^T)^T \\ &= A^T A\end{aligned}$$

Thus $A^T A$ is indeed symmetric.