Problem Set #4

(0.4.9) Evaluate $(f \circ g \circ h)(x)$ at $x = a$ for the following

(a) $f(x) = x^2 - 1$, $g(x) = 3x$, $h(x) = -x + 2$, for $a = 3$

$$
\begin{aligned}
(f \circ g \circ h)(a) = f(g(h(3))) &= f(g(-3+2)) \\
&= f(g(-1)) \\
&= f(-3) \\
&= (-3)^2 - 1 \\
&= 8
\end{aligned}
$$

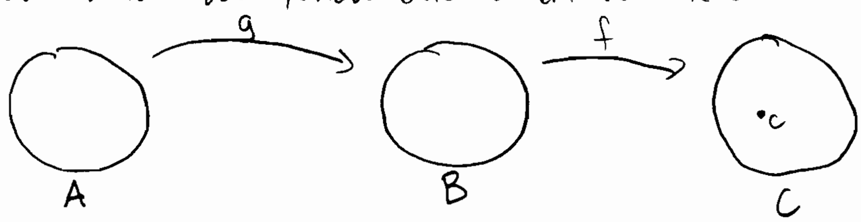
(b) $f(x) = x^2$, $g(x) = x - 3$, $h(x) = x - 3$, for $a = 1$

$$
\begin{aligned}
(f \circ g \circ h)(x) = f(g(h(1))) &= f(g(1-3)) \\
&= f(g(-2)) \\
&= f(-2-3) \\
&= f(-5) \\
&= (-5)^2 \\
&= 25
\end{aligned}
$$

(0.4.10) (a) Prove Proposition 0.4.16.

Let the functions $f: B \to C$ and $g: A \to B$ be onto. Then the composition $(f \circ g)$ is onto.

Let's spell out what we know and what we need to show




The function $f: B \to C$ is onto, so for every $c \in C$, there is a $b \in B$ such that $f(b) = c$. Note that I am not claiming that $b$ is unique, merely that it must exist. Likewise, $g: A \to B$ is onto, so a similar statement applies

We want to show that $f \circ g: A \to C$ is onto, so we must show that for any $c \in C$, there is an $a \in A$ such that $(f \circ g)(a) = c$.

Then let $c \in C$. Since $f$ is onto, there is a $b \in B$ such that $f(b) = c$. For the $b$, the fact that $g$ is onto means that there is an $a \in A$ such that $g(a) = b$. I claim that this is tle $a$ we want, since

$$(f \circ g)(a) = f(g(a)) = f(b) = c.$$

(b) Prove Proposition 0.4.17:

Let $f: B \to C$ and $g: A \to B$ be one to one. Then the composition $(f \circ g)$ is one to one.

Saying that $f: B \to C$ is one-to-one means that for every $c \in C$, there is at most one $b \in B$ such that $f(b) = c$. This is equivalent to saying that if $f(b) = f(b')$, then $b = b'$. We have a similar definition for $g$.

We want to show that $(f \circ g)$ is one-to-one, so I will show that if $(f \circ g)(a) = (f \circ g)(a')$, then $a = a'$.

Suppose that $(f \circ g)(a) = (f \circ g)(a')$ for some $a, a' \in A$. Then $f(g(a)) = f(g(a'))$, so $g(a) = g(a')$ since $f$ is one to one. But then $g(a) = g(a')$ implies that $a = a'$ since $g$ is one to one. Thus $(f \circ g)(a) = (f \circ g)(a')$ implies that $a = a'$, which is equivalent to the book's definition of $(f \circ g)$ being one-to-one.

(1.4.3) Normalize the following vectors.

(a) $\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \vec{v}$ $\quad |\vec{v}| = \sqrt{0^2+1^2+4^2} = \sqrt{17}$ $\quad$ So normalized vector is $\frac{1}{\sqrt{17}}\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{bmatrix}$

(b) $\vec{v} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ $\quad |\vec{v}| = \sqrt{(-3)^2+7^2} = \sqrt{58}$ $\quad$ Normalized vector is $\frac{1}{\sqrt{58}}\begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{-3}{\sqrt{58}} \\ \frac{7}{\sqrt{58}} \end{bmatrix}$

(c) $\vec{v} = \begin{bmatrix} \sqrt{2} \\ -2 \\ -5 \end{bmatrix}$ $\quad |\vec{v}| = \sqrt{(\sqrt{2})^2+(-2)^2+(-5)^2} = \sqrt{31}$ $\quad$ Normalized vector is $\frac{1}{\sqrt{31}}\begin{bmatrix} \sqrt{2} \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{31}} \\ \frac{-2}{\sqrt{31}} \\ \frac{-5}{\sqrt{31}} \end{bmatrix}$

(1.4.5) Calculate the angles between the following pairs of vectors.

Recall that $\cos(\alpha) = \dfrac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$ where $\alpha$ is the angle between $\vec{v}$ and $\vec{w}$ with $0 \le \alpha \le \pi$.

(a) $\begin{array}{cc} \vec{v} & \vec{w} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{array}$

$\vec{v}\cdot\vec{w} = 1+0+0 = 1$

$|\vec{v}| = \sqrt{1^2+0^2+0^2} = 1$

$|\vec{w}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

so $\cos(\alpha) = \frac{1}{\sqrt{3}}$

$\alpha \cong 54.7^\circ$ or $.955$ radians

(b) $\begin{array}{cc} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ \vec{v} & \vec{w} \end{array}$

$\vec{v}\cdot\vec{w} = 1+0+(-1)+0 = 0$

$|\vec{v}| = \sqrt{1^2+0^2+(-1)^2+0^2} = \sqrt{2}$

$|\vec{w}| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$

so $\cos(\alpha) = 0$

$\alpha = 90^\circ$ or $\pi/2$ radians

(1.4.7) For each of the following matrices, compute its determinant and its inverse if it exists

Recall that for a $2\times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det A = ad - bc$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(provided $ad - bc \neq 0$)

(a) $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \det A = 2\cdot 0 - (-1)(1) = 1$ $\quad A^{-1} = \frac{1}{1}\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \det B = 1\cdot 1 - 1\cdot 1 = 0$ $\quad B^{-1}$ does not exist

(c) $C = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ $\quad \det C = ad - 0b = ad$ $\quad C^{-1} = \frac{1}{ad}\begin{bmatrix} d & -b \\ 0 & a \end{bmatrix}$, provided $ad \neq 0$.

(d) $D = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ $\quad \det D = (1)(-1) - (1)(-1) = 0$ $\quad D^{-1}$ does not exist

(1.4.8) Compute the determinants of the matrices in the margin at left.

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

$$\det A = 1 \cdot \det\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - (-1)\det\begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} + 2\det\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= 0 + (4-6) + 2(2-3)$$

$$= -4$$

(b) $B = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$

$$\det B = a \cdot \det\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} - 0 \cdot \det\begin{bmatrix} b & c \\ 0 & f \end{bmatrix} + 0 \cdot \det\begin{bmatrix} b & c \\ d & e \end{bmatrix}$$

$$= adf + 0 + 0$$

$$= adf$$

(c) $C = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & g \end{bmatrix}$

$$\det C = a \cdot \det\begin{bmatrix} d & 0 \\ f & g \end{bmatrix} - c \cdot \det\begin{bmatrix} b & 0 \\ f & g \end{bmatrix} + e \cdot \det\begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}$$

$$= a(dg-0) - c(bg-0) + e(0-0)$$

$$= adg - cbg$$

(1.4.10)

(a) Let $A$ be a matrix. Show that $|A^k| \leq |A|^k$. Compute $|A^3|$ and $|A|^3$ for $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

We know from Proposition 1.4.11 that $|BC| \leq |B||C|$ for any matrices $B$ and $C$ for which the product $BC$ makes sense. Letting $B = A$ and $C = A$, this tells us that $|A^2| \leq |A|^2$. We will use this fact to prove the claim by induction.

Base case: $k = 1$

This case is obvious since $|A^1| = |A| = |A|^1$.

Inductive step: Assume that $|A^k| \leq |A|^k$. We will show that $|A^{k+1}| \leq |A|^{k+1}$.

Note that $A^{k+1} = (A^k) \cdot A$. Then

$$|A^{k+1}| = |A^k \cdot A| \underset{\substack{\uparrow \\ \text{by Prop. 1.4.11}}}{\leq} |A^k| \cdot |A| \underset{\substack{\uparrow \\ \text{by inductive hypothesis}}}{\leq} |A|^k \cdot |A| = |A|^{k+1}$$

This completes the proof.

For the second part,

$$\begin{array}{ccc} & \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} & \underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}}_{A^2} & \underbrace{\begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix}}_{A^3} \end{array}$$

thus $|A^3| = \sqrt{5^2 + 7^2 + 7^2 + 10^2}$

$= \sqrt{223}$ (slightly less than 15)

Meanwhile, $|A| = \sqrt{1^2 + 1^2 + 1^2 + 2^2} = \sqrt{7}$, so

$$|A|^3 = (\sqrt{7})^3 = \sqrt{7} \cdot \sqrt{7} \cdot \sqrt{7} = \sqrt{7 \cdot 7 \cdot 7} = \sqrt{343}.$$

(b) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$. Without computations, explain why the following are true or false:

(i) $|\vec{u} \cdot \vec{v}| < |\vec{u}||\vec{v}|$

Schwarz's inequality says that $|\vec{u} \cdot \vec{v}| \leq |\vec{u}||\vec{v}|$ with equality if and only if one of $\vec{u}, \vec{v}$ is a multiple of the other. Since $\vec{v} = -2\vec{u}$, this implies that $|\vec{u} \cdot \vec{w}| = |\vec{u}||\vec{v}|$ is true.

(ii) $|\vec{u} \cdot \vec{w}| = |\vec{u}||\vec{w}|$

Note that $\vec{u}$ and $\vec{w}$ are not multiples of each other, so by Schwarz's inequality the claim is false.

(continues) →

(1.4.10 continued)

(c) Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be two vectors such that $v_1 w_2 < v_2 w_1$. Does $\vec{w}$ lie clockwise or counterclockwise from $\vec{v}$?

From Proposition 1.4.14 (2) we know that $\det[\vec{a}, \vec{b}]$ is positive if and only if $\vec{a}$ is clockwise from $\vec{b}$. It is negative if and only if $\vec{a}$ lies counterclockwise from $\vec{b}$.

Note that $v_1 w_2 < v_2 w_1$ means $v_1 w_2 - v_2 w_1 < 0$. But $v_1 w_2 - v_2 w_1 = \det[\vec{v}, \vec{w}]$. Thus $\det[\vec{v}, \vec{w}]$ is negative, so $\vec{v}$ lies counterclockwise from $\vec{w}$, so $\vec{w}$ lies clockwise from $\vec{v}$.

(1.4.13) Show that the cross product of two vectors pointing in the same direction is zero.

When we say that the cross product is zero, recall that the cross product is a vector, so we are really being asked to show that the cross product is $\vec{0}$.

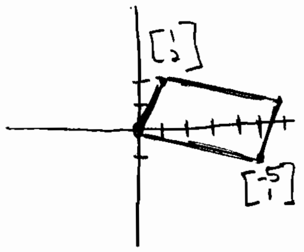
Two vectors point in the same direction precisely when one is a multiple of the other. Therefor let our two vectors be

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{and} \quad \alpha\vec{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \end{bmatrix}$$

Then $\vec{v} \times \alpha\vec{v} = \begin{bmatrix} v_2 \alpha v_3 - v_3 \alpha v_2 \\ -v_1 \alpha v_3 + v_3 \alpha v_1 \\ v_1 \alpha v_2 - v_2 \alpha v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$ So the proof is complete.

(1.4.16) What is the area of the parallelogram with vertices at

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \end{pmatrix}?$$




This is the parallelogram spanned the vectors $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

By Proposition 1.4.14, the area of this parallelogram is equal to $|\det[\vec{a}, \vec{b}]|$

Thus $\text{Area} = |\det[\vec{a}, \vec{b}]| = \left| \det \begin{bmatrix} 1 & -5 \\ 2 & 1 \end{bmatrix} \right| = |1 \cdot 1 - (-5)(2)| = |11| = 11.$

(1.4.19) (a) What is the length of $\vec{v}_n = \vec{e}_1 + \cdots + \vec{e}_n \in \mathbb{R}^n$.

First recall that for any vector $\vec{w}$, the length of $\vec{w}$ is $|\vec{w}| = \sqrt{\vec{w}\cdot\vec{w}}$. Next observe that

$$\vec{e}_i \cdot \vec{e}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Thus
$$\begin{aligned} \vec{v}_n \cdot \vec{v}_n &= (\vec{e}_1 + \cdots + \vec{e}_n) \cdot (\vec{e}_1 + \cdots + \vec{e}_n) \\ &= \vec{e}_1\cdot\vec{e}_1 + \cdots + \vec{e}_n \cdot \vec{e}_n \qquad \text{(other terms are 0)} \\ &= \underbrace{1 + \cdots + 1}_{n \text{ times}} \\ &= n \end{aligned}$$

Therefore $|\vec{v}_n| = \sqrt{n}$

(b) what is the angle $\alpha_n$ between $\vec{v}_n$ and $\vec{e}_1$? What is $\lim_{n\to\infty} \alpha_n$?

Let $\alpha_n$ be the angle between $\vec{v}_n$ and $\vec{e}_1$. Then by Definition 1.4.6,

$$\cos(\alpha_n) = \frac{\vec{v}_n \cdot \vec{e}_1}{|\vec{v}_n|\,|\vec{e}_1|}$$

We know from part (a) that $|\vec{v}_n| = \sqrt{n}$. Also,

$$\vec{v}_n \cdot \vec{e}_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1 \qquad \text{and} \quad |\vec{e}_1| = \text{square root of } \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sqrt{1} = 1$$

Thus $\cos(\alpha_n) = \frac{1}{\sqrt{n}}$, so $\alpha_n = \arccos\left(\frac{1}{\sqrt{n}}\right)$

Taking the limit as $n \to \infty$ of both sides, we get

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \arccos\left(\tfrac{1}{\sqrt{n}}\right) = \arccos\left(\lim_{n\to\infty} \tfrac{1}{\sqrt{n}}\right) = \arccos(0)$$
$$= 90^\circ \text{ or } \pi/2 \text{ radians}$$

I need to know that arccos is continuous, at least in the region $(0,1)$ of the domain, in order to pull the limit inside the arccosine.

(1.4.24) Let $\vec{v} \in \mathbb{R}^n$ be a nonzero vector, and denote by $\vec{v}^\perp \subset \mathbb{R}^n$ the set of vectors $\vec{w} \in \mathbb{R}^n$ such that $\vec{v} \cdot \vec{w} = 0$, so $\vec{v}^\perp = \{ \vec{w} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \}$.

(a) Show that $\vec{v}^\perp$ is a subspace of $\mathbb{R}^n$.

We must show that $\vec{v}^\perp$ is closed under addition and scalar multiplication.
Let $\alpha \in \mathbb{R}$ and $\vec{w}_1, \vec{w}_2 \in \vec{v}^\perp$, so that $\vec{v} \cdot \vec{w}_1 = \vec{v} \cdot \vec{w}_2 = 0$. Then

$$\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = \vec{v} \cdot \vec{w}_1 + \vec{v} \cdot \vec{w}_2 = 0 + 0 = 0 \qquad \text{so } (\vec{w}_1 + \vec{w}_2) \in \vec{v}^\perp$$

$$\vec{v} \cdot (\alpha \vec{w}_1) = \alpha (\vec{v} \cdot \vec{w}_1) = \alpha \cdot 0 = 0 \qquad \text{so } \alpha \vec{w}_1 \in \vec{v}^\perp$$

Thus $\vec{v}^\perp$ is closed under addition and scalar multiplication, so it is a subspace of $\mathbb{R}^n$.

(b) Given any vector $\vec{a} \in \mathbb{R}^n$, show that $\vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$ is an element of $\vec{v}^\perp$.

All that we need to show is that

$$\vec{v} \cdot \left( \vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} \right) = 0$$

Note that $\frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2}$ is a constant and that $\vec{v} \cdot \vec{v} = |\vec{v}|^2$ by definition. Then

$$\begin{aligned} \vec{v} \cdot \left( \vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} \right) &= \vec{v} \cdot \vec{a} - \frac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} (\vec{v} \cdot \vec{v}) \\ &= \vec{v} \cdot \vec{a} - \vec{a} \cdot \vec{v} \\ &= 0 \end{aligned}$$

(c) Define the projection of $\vec{a}$ onto $\vec{v}^\perp$ by the formula

$$P_{\vec{v}^\perp}(\vec{a}) = \vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$$

Show that there is a unique number $t(\vec{a})$ such that $(\vec{a} + t(\vec{a}) \vec{v}) \in \vec{v}^\perp$, and show that $\vec{a} + t(\vec{a}) \vec{v} = P_{\vec{v}^\perp}(\vec{a})$.

We already know that $t(\vec{a}) = -\frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2}$ works from our work in part (b). Now to show that this is the unique such $t(\vec{a})$, suppose that $(\vec{a} + t(\vec{a}) \vec{v}) \in \vec{v}^\perp$, so that $(\vec{a} + t(\vec{a}) \vec{v}) \cdot \vec{v} = 0$. Then

$$\begin{aligned} \vec{a} \cdot \vec{v} + t(\vec{a}) \vec{v} \cdot \vec{v} &= 0 \\ t(\vec{a}) \vec{v} \cdot \vec{v} &= -\vec{a} \cdot \vec{v} \\ t(\vec{a}) &= \frac{-\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \end{aligned}$$

Thus this is the only solution. Plugging in for $t(\vec{a})$, we do indeed find that

$$\vec{a} + t(\vec{a}) \vec{v} = \vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = P_{\vec{v}^\perp}(\vec{a}).$$

(1.4.26) Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$

(a) What is the angle $\alpha\binom{x}{y}$ between $\begin{bmatrix} x \\ y \end{bmatrix}$ and $A\begin{bmatrix} x \\ y \end{bmatrix}$?

Note that $\begin{bmatrix} x \\ y \end{bmatrix}$ and $A\begin{bmatrix} x \\ y \end{bmatrix}$ are both vectors so we may ask for the angle between them, but this angle will depend on $\begin{bmatrix} x \\ y \end{bmatrix}$, hence the notation $\alpha\begin{bmatrix} x \\ y \end{bmatrix}$.

Explicitly, $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ 3x+4y \end{bmatrix}$.

For general vectors $\vec{v}$ and $\vec{w}$, $\alpha = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}\right)$, where $\alpha$ is the angle between them.

For us, let $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} x-2y \\ 3x+4y \end{bmatrix}$. Then

$$|\vec{v}| = \sqrt{x^2+y^2}$$
$$|\vec{w}| = \sqrt{(x-2y)^2+(3x+4y)^2} = \sqrt{10x^2+20xy+20y^2}$$
$$\vec{v}\cdot\vec{w} = x^2-2xy+3xy+4y^2 = x^2+xy+4y^2$$

Thus $\alpha\binom{x}{y} = \arccos\left(\frac{x^2+xy+4y^2}{\sqrt{x^2+y^2}\cdot\sqrt{10x^2+20xy+20y^2}}\right)$

I didn't find any way to simplify this nicely, so I'll leave it in this form.

(b) Is there any nonzero vector $\begin{bmatrix} x \\ y \end{bmatrix}$ that is rotated by $\pi/2$?

If we plug in $\alpha\binom{x}{y} = \pi/2$ and take the cosine of both sides, then since $\cos(\pi/2) = 0$, we get

$$0 = \frac{x^2+xy+4y^2}{\sqrt{x^2+y^2}\cdot\sqrt{10x^2+20xy+20y^2}}$$

so that $$0 = x^2+xy+4y^2$$

We can solve this in a number of ways. Using the quadratic formula (so solving for $x$ in terms of $y$) we get

$$x = \frac{-y \pm \sqrt{y^2-16y^2}}{2}$$

The term under the square root is negative for all $y$, so there are no solutions. Hence, no vector is rotated by the angle $\pi/2$ when $A$ is applied to it.