Problem Set #5

(2.1.2) Write each of the following systems of equations as a single matrix, and row reduce the matrix to echelon form.

(a)

$$\begin{aligned} 3y - z &= 0 \\ -2x + y + 2z &= 0 \\ x \quad -5x &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 & 3 & -1 & 0 \\ -2 & 1 & 2 & 0 \\ 1 & 0 & -5 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -5 & 0 \\ -2 & 1 & 2 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 23 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(b)

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ -2x_2 + x_3 &= 2 \\ x_1 \quad -2x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & -2 & 1 & 2 \\ 1 & 0 & -2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & -2 & 1 & 2 \\ 2 & 3 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & -2 & 1 & 2 \\ 0 & 3 & 3 & 3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -1/2 & -1 \\ 0 & 3 & 3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -1/2 & -1 \\ 0 & 0 & 9/2 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -1/2 & -1 \\ 0 & 0 & 1 & 4/3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 4/3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 4/3 \end{bmatrix}$$

(2.1.3) Bring the matrices in the margin to echelon form using row operations

(a) $$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&3\\0&-3&-6\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&3\\0&1&2\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&-1\\0&1&2\end{bmatrix}$$

(b) $$\begin{bmatrix}1&-1&1\\-1&0&2\\-1&1&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&-1&1\\0&-1&3\\0&0&2\end{bmatrix}\longrightarrow\begin{bmatrix}1&-1&1\\0&1&-3\\0&0&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

(c) $$\begin{bmatrix}1&2&3&5\\2&3&0&-1\\0&1&2&3\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&3&5\\0&-1&-6&-11\\0&1&2&3\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&3&5\\0&1&6&11\\0&1&2&3\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&3&5\\0&1&6&11\\0&0&-4&-8\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}1&2&3&5\\0&1&6&11\\0&0&1&2\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&0&-1\\0&1&0&-1\\0&0&1&2\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&2\end{bmatrix}$$

(d) $$\begin{bmatrix}1&3&-1&4\\1&2&1&2\\3&7&1&9\end{bmatrix}\longrightarrow\begin{bmatrix}1&3&-1&4\\0&-1&2&-2\\0&-2&4&-3\end{bmatrix}\longrightarrow\begin{bmatrix}1&3&-1&4\\0&1&-2&2\\0&-2&4&-3\end{bmatrix}\longrightarrow\begin{bmatrix}1&3&-1&4\\0&1&-2&2\\0&0&0&1\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}1&3&-1&0\\0&1&-2&0\\0&0&0&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&5&0\\0&1&-2&0\\0&0&0&1\end{bmatrix}$$

(e) $$\begin{bmatrix}1&1&1&1\\2&-3&3&3\\1&-4&2&2\end{bmatrix}\longrightarrow\begin{bmatrix}1&1&1&1\\0&-5&1&1\\0&-5&1&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&1&1&1\\0&-5&1&1\\0&0&0&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&1&1&1\\0&1&-1/5&-1/5\\0&0&0&0\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}1&0&6/5&6/5\\0&1&-1/5&-1/5\\0&0&0&0\end{bmatrix}$$

(2.1.5) Show that any row operation can be undone by another row operation. Note the importance of the word "nonzero" in definition 2.1.1 of row operations.

There are three types of row operations, as explained in definition 2.1.1. We must show how to undo each type.

(1) Multiplying a row by a nonzero number m.

Undo this by multiplying the row by $\frac{1}{m}$. Note that $\frac{1}{m}$ makes sense since $m \neq 0$.

(2) Adding k times row i to row j.

Undo this by adding $-k$ times row i to row j. Thus we have added a total of $k-k=0$ times row i to row j, and row i was never changed.

(3) Exchange two rows.

Undo this by exchanging the same two rows again.

(2.1.8) Show that if A is square, and $\tilde{A}$ is A row reduced to echelon form, then either $\tilde{A}$ is the identity, or the last row is a row of zeros.

I will frequently refer to the properties of echelon form given in Definition 2.1.4, though I will not restate them here.

First I want to show that a matrix in echelon form can only have one pivot per row and one pivot per column. By property (1) of echelon form, the first nonzero entry in each row is a pivotal 1. Obviously no row can have more than one first nonzero entry. For columns, property (3) of echelon form states that in a column with a pivotal 1, all other entries are 0. Thus there is at most one pivot per column as well.

Now A is a square matrix, say $n \times n$, so $\tilde{A}$ is also $n \times n$. Then by the above reasoning, $\tilde{A}$ has at most n pivots. I will split into two cases: when $\tilde{A}$ has exactly n pivots, and when $\tilde{A}$ has less than n pivots.

Case 1: Suppose that the $n \times n$ echelon form matrix $\tilde{A}$ has exactly n pivots. Property (2) of echelon form states that the pivotal 1 of a lower row is always to the right of a pivotal 1 in a higher row. Every column must have a pivot, so each pivot must lie exactly one entry down and one to the right of the pivot in the row above. With n such pivots in an $n \times n$ matrix, this means 1's on the main diagonal. Since every other entry in a pivotal column (all of them) must be 0, this means $\tilde{A}$ is the $n \times n$ identity.

Case 2: Suppose that the $n \times n$ echelon form matrix $\tilde{A}$ has less than n pivots. Then there must be a row without a pivot. Pivots occur as the first nonzero entry in a row in an echelon form matrix, so this means that $\tilde{A}$ has a row of zeros. By property (4) of echelon form, this row of zeros is at the bottom of $\tilde{A}$, so the proof is complete.