Problem Set 7.

(2.4.2) (a)

Do the vectors
$$\vec{w}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix}-2\\1\\2\end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$$
form a basis of $\mathbb{R}^3$? If so, is this basis orthogonal?

A basis of $\mathbb{R}^3$ would need to span $\mathbb{R}^3$ and be linearly independent. We can check both of these properties via row reduction of the matrix whose columns are $\vec{w}_1, \vec{w}_2, \vec{w}_3$. See Theorem 2.2.2 for interpretation.

$$\begin{bmatrix}1 & -2 & -1\\2 & 1 & 1\\3 & 2 & -1\end{bmatrix} \to \begin{bmatrix}1 & -2 & -1\\0 & 5 & 3\\0 & 8 & 2\end{bmatrix} \to \begin{bmatrix}1 & -2 & -1\\0 & 1 & 3/5\\0 & 4 & 1\end{bmatrix} \to \begin{bmatrix}1 & -2 & -1\\0 & 1 & 3/5\\0 & 0 & -7/5\end{bmatrix}$$

$$\to \begin{bmatrix}1 & -2 & -1\\0 & 1 & 3/5\\0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

The echelon form has no rows of zeros, so $\vec{w}_1, \vec{w}_2$, and $\vec{w}_3$ span $\mathbb{R}^3$, and every column has a pivot, so the three vectors are linearly independent. Thus these vectors do form a basis of $\mathbb{R}^3$.

To check orthogonality, we simply compute the dot product of each pair of vectors in our basis. If each dot product is zero then the basis is orthogonal.

$$\vec{w}_1 \cdot \vec{w}_2 = \begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}-2\\1\\2\end{bmatrix} = 6$$

We don't even need to compute the other dot products. This one is nonzero, so our basis is not orthogonal.

(b) Is $\begin{bmatrix}4\\1\\2\end{bmatrix}$ in span$\left(\begin{bmatrix}4\\2\\1\end{bmatrix}, \begin{bmatrix}3\\0\\4\end{bmatrix}, \begin{bmatrix}2\\1\\4\end{bmatrix}\right)$? In span$\left(\begin{bmatrix}4\\2\\1\end{bmatrix}, \begin{bmatrix}3\\0\\4\end{bmatrix}, \begin{bmatrix}5\\1\\4.5\end{bmatrix}\right)$?

We check these questions using the method in Example 2.4.4. (Next page)

(continued) →

(2.4.2 (b) continued)

For the first span, we row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 4 & 3 & 2 & 4 \\ 2 & 0 & 1 & 1 \\ 1 & 4 & 4 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 4 & 2 \\ 2 & 0 & 1 & 1 \\ 4 & 3 & 2 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 4 & 2 \\ 0 & -8 & -7 & -3 \\ 0 & -13 & -14 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 4 & 2 \\ 0 & 1 & 7/8 & 3/8 \\ 0 & -13 & -14 & -4 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 4 & 2 \\ 0 & 1 & 7/8 & 3/8 \\ 0 & 0 & -21/8 & 7/8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 4 & 2 \\ 0 & 1 & 7/8 & 3/8 \\ 0 & 0 & 1 & -1/3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 0 & 2/3 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4/3 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/3 \end{array}\right]$$

Row reduction shows that there is a solution, so the answer to the first span question is <u>yes</u>.

We repeat the procedure for the second span question:

$$\left[\begin{array}{ccc|c} 4 & 3 & 5 & 4 \\ 2 & 0 & 1 & 1 \\ 1 & 4 & 4.5 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 4.5 & 2 \\ 2 & 0 & 1 & 1 \\ 4 & 3 & 5 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 4.5 & 2 \\ 0 & -8 & -8 & -3 \\ 0 & -13 & -13 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 4.5 & 2 \\ 0 & 1 & 1 & 3/8 \\ 0 & 1 & 1 & 4/13 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 4.5 & 2 \\ 0 & 1 & 1 & 3/8 \\ 0 & 0 & 0 & -7/104 \end{array}\right]$$

At this point I'll stop the row reduction since the last row has an impossible equation: $0 = -7/104$. Thus the answer to the span question is <u>no</u>.

(2.4.3) The vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ form an orthogonal basis of $\mathbb{R}^2$. Use these vectors to create an orthonormal basis for $\mathbb{R}^2$.

Since our basis is already orthogonal, all we need for orthogonality is for each basis vector to have length 1. We do this by normalizing the vectors (dividing by their lengths). Remember that the length of a vector $\vec{v}$ is $\sqrt{\vec{v} \cdot \vec{v}}$, so each of our vectors has length $\sqrt{2}$. Thus our orthonormal basis is

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

(2.4.5) Show that $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$ and is the smallest subspace containing $\vec{v}_1, \dots, \vec{v}_k$.

First we will show that $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$ by demonstrating closure under addition and scalar multiplication.

Let $\vec{a}, \vec{b} \in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$ so that

$$\vec{a} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_k\vec{v}_k$$
$$\vec{b} = y_1\vec{v}_1 + y_2\vec{v}_2 + \dots + y_k\vec{v}_k \qquad \text{for some } x_1, \dots, x_k, y_1, \dots, y_k \in \mathbb{R}.$$

Then $\vec{a}+\vec{b} = (x_1+y_1)\vec{v}_1 + (x_2+y_2)\vec{v}_2 + \dots + (x_k+y_k)\vec{v}_k$, so $(\vec{a}+\vec{b}) \in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$.

Let $c \in \mathbb{R}$ be a scalar. Then

$$c\vec{a} = c(x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_k\vec{v}_k) = (cx_1)\vec{v}_1 + (cx_2)\vec{v}_2 + \dots + (cx_k)\vec{v}_k$$

Thus $(c\vec{a}) \in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$ as well, so $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$ is closed under scalar multiplication and addition, and hence is a subspace of $\mathbb{R}^n$

Next we need to show that $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$ is the smallest subspace containing $\vec{v}_1, \dots, \vec{v}_k$. We do this by showing that if W is any other subspace of $\mathbb{R}^n$ such that $\vec{v}_1, \dots, \vec{v}_k \in W$, then $\text{span}(\vec{v}_1, \dots, \vec{v}_k) \subseteq W$, so that any other subspace W with the desired property is larger than $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$. Thus $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$ will be the smallest.

Let W be a subspace of $\mathbb{R}^n$ with $\vec{v}_1, \dots, \vec{v}_k \in W$. Let $\vec{a} \in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$, so that $\vec{a} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_k\vec{v}_k$ for some scalars $x_1, \dots, x_k$. Then $\vec{a} \in W$ as well since $\vec{v}_1, \dots, \vec{v}_k \in W$ and W is closed under scalar multiplication and addition (by virtue of being a subspace). Thus every element of $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$ is an element of W, so $\text{span}(\vec{v}_1, \dots, \vec{v}_k) \subseteq W$, so $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$ is the smallest subspace of $\mathbb{R}^n$ containing $\vec{v}_1, \dots, \vec{v}_k$.

(2.4.7) Let A be an $n \times n$ matrix. Show that A is orthogonal if and only if $A^T A = I$.

First, a word of caution: saying that a matrix is orthogonal means that the columns of A are orthonormal. This is terrible terminology, but we'll deal with it.

This is an if and only if proof, so we need to prove both directions I begin with some general observations. Let $B = A^T A$, so that b_{ij} denotes the entry in the i^{th} row and j^{th} column of B. Then b_{ij} is the dot product of the i^{th} row of A^T with the j^{th} column of A By definition of the transpose, this means that b_{ij} is the dot product of the i^{th} column of A with the j^{th} column of A.

Now assume that A is orthogonal, so that its columns are orthonormal. Then the dot product of the i^{th} column with the j^{th} column is 1 if $i=j$ and 0 if $i \neq j$. Hence $b_{ij}=1$ if $i=j$ (on the diagonal) and $b_{ij}=0$ if $i \neq j$ (off the diagonal). Thus $B = A^T A = I$.

Conversely, assume that $B = A^T A = I$ Then $b_{ij}=1$ if $i=j$ and 0 otherwise. Hence the dot product of the i^{th} column of A with the j^{th} column of A is 1 if $i=j$, and 0 otherwise. This is precisely the definition of A being orthogonal, so the proof is complete.

(2.4.10) Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1\\3\end{bmatrix}$. Let x and y be the coordinates with respect to the standard basis $\{\vec{e}_1, \vec{e}_2\}$ and let u and v be the coordinates with respect to $\{\vec{v}_1, \vec{v}_2\}$. Write the equations to translate from (x,y) to (u,v) and back. Use the equations to write the vector $\begin{bmatrix}3\\-5\end{bmatrix}$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

The notation above says that if $\vec{w} \in \mathbb{R}^2$, then $\vec{w} = x\vec{e}_1 + y\vec{e}_2$, where x and y depend on the specific $\vec{w}$. This is what is meant by the coordinates with respect to the standard basis. Similarly, $\vec{w} = u\vec{v}_1 + v\vec{v}_2$. Thus,

$$x\vec{e}_1 + y\vec{e}_2 = u\vec{v}_1 + v\vec{v}_2$$

and hence

$$x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} = u\begin{bmatrix}1\\1\end{bmatrix} + v\begin{bmatrix}1\\3\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}u+v\\u+3v\end{bmatrix}$$

Thus $x = u+v$ and $y = u+3v$, so we have the equations to translate from (u,v) to (x,y). To go backwards, note that we can rewrite this using matrices as

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} \qquad (*)$$

Then to go back, we need to invert the 2×2 matrix in $(*)$.

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\1 & 3 & 0 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & 2 & -1 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & 1 & -\frac{1}{2} & \frac{1}{2}\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 0 & \frac{3}{2} & -\frac{1}{2}\\0 & 1 & -\frac{1}{2} & \frac{1}{2}\end{array}\right]$$

Thus the inverse of $\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}$ is $\begin{bmatrix}\frac{3}{2} & -\frac{1}{2}\\-\frac{1}{2} & \frac{1}{2}\end{bmatrix}$, so multiplying both sides of $(*)$ on the left by the inverse, we get

$$\begin{bmatrix}\frac{3}{2} & -\frac{1}{2}\\-\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}u\\v\end{bmatrix}.$$

Hence we have the equations $u = \frac{3}{2}x - \frac{1}{2}y$ and $v = -\frac{1}{2}x + \frac{1}{2}y$ to translate (x,y) back to (u,v).

For the last part of the problem, note that $\begin{bmatrix}3\\-5\end{bmatrix} = 3\vec{e}_1 - 5\vec{e}_2$, so $x=3$ and $y=-5$. Plugging this into our equations above, we get

$$u = \tfrac{3}{2}(3) - \tfrac{1}{2}(-5) = 7$$

$$v = -\tfrac{1}{2}(3) + \tfrac{1}{2}(-5) = -4$$

Hence $\begin{bmatrix}3\\-5\end{bmatrix} = 7\vec{v}_1 - 4\vec{v}_2$, which you may check explicitly if you like.

(2.4.11) Suppose we want to approximate $\int_0^1 f(x)\,dx \approx \sum_{i=0}^{n} a_{i,n} f(\frac{i}{n})$ (1)

(a) For $n=1, n=2, n=3$, write the system of linear equations which the $a_{0,n}, \dots, a_{n,n}$ must satisfy so that formula (1) is an equality for each of the functions

$$f(x)=1,\ f(x)=x,\ f(x)=x^2,\ \dots,\ f(x)=x^n$$

We'll do the $n=1$ case first. For $n=1$, we only require equality for $f(x)=1$ and $f(x)=x$ (since the maximal power of x is $n=1$).

$\underline{n=1}$: For $f(x)=1$ in formula (1), we get

$$\int_0^1 f(x)\,dx = \int_0^1 1\,dx = 1$$

$$\sum_{i=0}^{n} a_{i,n} f(\tfrac{i}{n}) = a_{0,1} f(\tfrac{0}{1}) + a_{1,1} f(\tfrac{1}{1}) = a_{0,1} + a_{1,1}$$

This gives the equation $1 = a_{0,1} + a_{1,1}$

For $f(x)=x$ we get

$$\int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\sum_{i=0}^{n} a_{i,n} f(\tfrac{i}{n}) = a_{0,1} f(0) + a_{1,1} f(1) = a_{1,1}$$

This gives the equation $\frac{1}{2} = a_{1,1}$

Hence our system of equations is

$$\begin{aligned} a_{0,1} + a_{1,1} &= 1 \\ a_{1,1} &= \tfrac{1}{2}. \end{aligned}$$

Having done the $n=1$ case in detail, I will be more brief for the remaining cases:

$\underline{n=2}$:

$$f(x)=1 \Rightarrow \int_0^1 1\,dx = 1 = \sum_{i=0}^{2} a_{i,2} f(\tfrac{i}{2}) = a_{0,2} + a_{1,2} + a_{2,2}$$

$$f(x)=x \Rightarrow \int_0^1 x\,dx = \tfrac{1}{2} = \sum_{i=0}^{2} a_{i,2} f(\tfrac{i}{2}) = \tfrac{1}{2} a_{1,2} + a_{2,2}$$

$$f(x)=x^2 \Rightarrow \int_0^1 x^2\,dx = \tfrac{1}{3} = \sum_{i=0}^{2} a_{i,2} f(\tfrac{i}{2}) = \tfrac{1}{4} a_{1,2} + a_{2,2}$$

Thus our system of equations is

$$\begin{aligned} a_{0,2} + a_{1,2} + a_{2,2} &= 1 \\ \tfrac{1}{2} a_{1,2} + a_{2,2} &= \tfrac{1}{2} \\ \tfrac{1}{4} a_{1,2} + a_{2,2} &= \tfrac{1}{3} \end{aligned}$$

$\underline{n=3}$

$$f(x)=1 \Rightarrow \int_0^1 1\,dx = 1 = \sum_{i=0}^{3} a_{i,3} f(\tfrac{i}{3}) = a_{0,3} + a_{1,3} + a_{2,3} + a_{3,3}$$

$$f(x)=x \Rightarrow \int_0^1 x\,dx = \tfrac{1}{2} = \sum_{i=0}^{3} a_{i,3} f(\tfrac{i}{3}) = \tfrac{1}{3} a_{1,3} + \tfrac{2}{3} a_{2,3} + a_{3,3}$$

$$f(x)=x^2 \Rightarrow \int_0^1 x^2\,dx = \tfrac{1}{3} = \sum_{i=0}^{3} a_{i,3} f(\tfrac{i}{3}) = \tfrac{1}{9} a_{1,3} + \tfrac{4}{9} a_{2,3} + a_{3,3}$$

$$f(x)=x^3 \Rightarrow \int_0^1 x^3\,dx = \tfrac{1}{4} = \sum_{i=0}^{3} a_{i,3} f(\tfrac{i}{3}) = \tfrac{1}{27} a_{1,3} + \tfrac{8}{27} a_{2,3} + a_{3,3}$$

(continued) $\rightarrow$

(2.4.11 (a) continued)

Thus in the $n=2$ case our system of equations is

$$
\begin{aligned}
a_{0,3} + a_{1,3} + a_{2,3} + a_{3,3} &= 1 \\
\tfrac{1}{3}a_{1,3} + \tfrac{2}{3}a_{2,3} + a_{3,3} &= \tfrac{1}{2} \\
\tfrac{1}{9}a_{1,3} + \tfrac{4}{9}a_{2,3} + a_{3,3} &= \tfrac{1}{3} \\
\tfrac{1}{27}a_{1,3} + \tfrac{8}{27}a_{2,3} + a_{3,3} &= \tfrac{1}{4}
\end{aligned}
$$

(b) Solve these equations, and use them to give three appoximations of $\int_0^1 \frac{1}{x+1}\,dx$.

<u>n=1</u> We solve the $n=1$ system of equations by turning it into an augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \tfrac12 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \tfrac12 \\ 0 & 1 & \tfrac12 \end{array}\right] \qquad \text{Thus } a_{0,1} = \tfrac12, \quad a_{1,1} = \tfrac12$$

Using these coefficients in the first approximation of $\int_0^1 \frac{1}{x+1}dx$, we get

$$\int_0^1 \frac{1}{x+1}dx \approx \sum_{i=0}^{1} a_{i,1} \frac{1}{\left(\frac{i}{1}\right)+1} = \frac12\cdot\left(\frac{1}{0+1}\right) + \frac12\left(\frac{1}{1+1}\right) = \frac34$$

<u>n=2</u>: We proceed similarly:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \tfrac12 & 1 & \tfrac12 \\ 0 & \tfrac14 & 1 & \tfrac13 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 4 & \tfrac43 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & \tfrac13 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \tfrac16 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & \tfrac56 \\ 0 & 1 & 0 & \tfrac23 \\ 0 & 0 & 1 & \tfrac16 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \tfrac16 \\ 0 & 1 & 0 & \tfrac23 \\ 0 & 0 & 1 & \tfrac16 \end{array}\right]. \qquad \text{Thus } a_{0,2} = \tfrac16, \quad a_{1,2} = \tfrac23, \quad a_{2,2} = \tfrac16$$

Using these coefficients, we approximate

$$\int_0^1 \frac{1}{x+1}dx \approx \sum_{i=0}^{2} a_{i,2}\frac{1}{\left(\frac{i}{2}\right)+1} = \frac16\left(\frac{1}{0+1}\right) + \frac23\left(\frac{1}{\frac12+1}\right) + \frac16\left(\frac{1}{1+1}\right) = \frac16 + \frac49 + \frac1{12} = \frac{25}{36}$$

<u>n=3</u>:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & \tfrac13 & \tfrac23 & 1 & \tfrac12 \\ 0 & \tfrac19 & \tfrac49 & 1 & \tfrac13 \\ 0 & \tfrac1{27} & \tfrac8{27} & 1 & \tfrac14 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & \tfrac32 \\ 0 & 1 & 4 & 9 & 3 \\ 0 & 1 & 8 & 27 & \tfrac{27}{4} \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & \tfrac32 \\ 0 & 0 & 2 & 6 & \tfrac32 \\ 0 & 0 & 6 & 24 & \tfrac{21}{4} \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & \tfrac32 \\ 0 & 0 & 1 & 3 & \tfrac34 \\ 0 & 0 & 1 & 4 & \tfrac78 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & \tfrac32 \\ 0 & 0 & 1 & 3 & \tfrac34 \\ 0 & 0 & 0 & 1 & \tfrac18 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & \tfrac78 \\ 0 & 1 & 2 & 0 & \tfrac98 \\ 0 & 0 & 1 & 0 & \tfrac38 \\ 0 & 0 & 0 & 1 & \tfrac18 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \tfrac12 \\ 0 & 1 & 0 & 0 & \tfrac38 \\ 0 & 0 & 1 & 0 & \tfrac38 \\ 0 & 0 & 0 & 1 & \tfrac18 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \tfrac18 \\ 0 & 1 & 0 & 0 & \tfrac38 \\ 0 & 0 & 1 & 0 & \tfrac38 \\ 0 & 0 & 0 & 1 & \tfrac38 \end{array}\right]$$

Using these coefficients, we approximate

$$\int_0^1 \frac{1}{x+1}dx = \sum_{i=0}^{3} a_{i,3}\frac{1}{\left(\frac{i}{3}\right)+1} = \frac18\left(\frac{1}{0+1}\right) + \frac38\left(\frac{1}{\frac13+1}\right) + \frac38\left(\frac{1}{\frac23+1}\right) + \frac18\left(\frac{1}{1+1}\right) = \frac18 + \frac{9}{32} + \frac{9}{40} + \frac1{16}$$

$$= \frac{20}{160} + \frac{45}{160} + \frac{36}{160} + \frac{10}{160} = \frac{111}{160}$$

(2.4.12) Let $A_t = \begin{bmatrix} 2 & t \\ 0 & 2 \end{bmatrix}$

(a) Are the elements I, A_t, A_t^2, A_t^3 linearly independent in Mat(2,2)? Let $V_t = \text{span}\{I, A_t, A_t^2, A_t^3\}$. What is the dimension of V_t? (Answer depends on t).

I understand that Professor Meyer defined Mat(2,2) in class as the collection of all 2x2 matrices with real coefficients, which turns out to be a vector space of dimension 4.

Let's write out the matrices I, A_t, A_t^2, A_t^3.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_t = \begin{bmatrix} 2 & t \\ 0 & 2 \end{bmatrix}, \quad A_t^2 = \begin{bmatrix} 4 & 4t \\ 0 & 4 \end{bmatrix}, \quad A_t^3 = \begin{bmatrix} 8 & 12t \\ 0 & 8 \end{bmatrix}$$

With a little experimentation, we notice that

$$A_t^2 = 4A_t - 4I \quad \text{and} \quad A_t^3 = 12A_t - 16I$$

Furthermore, if $t=0$, then $A_t = 2I$. Thus the elements I, A_t, A_t^2, A_t^3 are definitely not linearly independent.

To find the dimension of V_t, we'll find a basis. Note that if $t=0$, then all of A_t, A_t^2, A_t^3 are multiples of I, so $\text{span}\{I, A_t, A_t^2, A_t^3\} = \text{span}\{I\}$. But $\{I\}$ is a basis for $\text{span}\{I\}$ since it spans by definition and is linearly independent. Thus our basis has one element, so when $t=0$, $\dim V_t = 1$.

When $t \neq 0$, I will show that $\text{span}\{I, A_t, A_t^2, A_t^3\} = \text{span}\{I, A_t\}$. We already know that $\text{span}\{I, A_t\} \subseteq \text{span}\{I, A_t, A_t^2, A_t^3\}$. Now let $B \in \text{span}\{I, A_t, A_t^2, A_t^3\}$, so that

$$B = b_1 I + b_2 A_t + b_3 A_t^2 + b_4 A_t^3$$
$$= b_1 I + b_2 A_t + (4b_3 A_t - 4b_3 I) + (12 b_4 A_t - 16 b_4 I).$$

Hence $B \in \text{span}\{I, A_t\}$, so by mutual containment,

$$\text{span}\{I, A_t\} = \text{span}\{I, A_t, A_t^2, A_t^3\}$$

I claim that I and A_t form a basis of $V_t = \text{span}\{I, A_t\}$. They span as we showed above, and are linearly independent when $t \neq 0$ since neither is a multiple of the other. Thus when $t \neq 0$, $\dim V_t = 2$.

(b) Show that the set W_t of matrices $B \in \text{Mat}(2,2)$ that satisfy $A_t B = B A_t$ is a subspace of Mat(2,2). What is its dimension? (Again depends on t

To show that W_t is a subspace, we must show that it is closed under addition and scalar multiplication. Let $B_1, B_2 \in W_t$ and $c \in \mathbb{R}$ be a scalar.

(continued →)

(2.4.12 (b) continued)

Then all we need to show is that $(B_1+B_2)A_t = A_t(B_1+B_2)$ so that $(B_1+B_2) \in W_t$, and that $(cB_1)A_t = A_t(cB_1)$ so that $(cB_1) \in W_t$ Note that

$$(B_1+B_2)A_t = B_1A_t + B_2A_t = A_tB_1 + A_tB_2 = A_t(B_1+B_2)$$

and $$(cB_1)A_t = c(B_1A_t) = c(A_tB_1) = A_t(cB_1)$$

Thus W_t is a subspace of $Mat(2,2)$

For the dimension, we begin the case $t=0$. In that case, note that $A_t = 2I$. So when we ask which B satisfy $BA_t = A_tB$, we're really asking which B commute with I. But all 2×2 matrices commute with the identity, so $W_t = Mat(2,2)$, and hence W_t has dimension 4.

When $t \neq 0$, we want to see which B satisfy $BA_t = A_tB$. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_t$, so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & t \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & t \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Evaluating both sides we get

$$\begin{bmatrix} 2a & at+2b \\ 2c & ct+2d \end{bmatrix} = \begin{bmatrix} 2a+ct & 2b+dt \\ 2c & 2d \end{bmatrix}$$

We could sit down and create a system of 4 equations and 4 unknowns to solve this, but I'll save some time and note that $c=0$ and $a=d$. Thus our matrix B must be of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

Matrices of this form are spanned by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which are also linearly independent (neither is a multiple of the other), so this is a basis of W_t, and hence $\dim W_t = 2$.

(c) Show that $V_t \subseteq W_t$. For what values of t are they equal?

Let $M \in V_t$. We will show that $M \in W_t$ as well so that $V_t \subseteq W_t$ Note that we showed earlier that $V_t = \text{span}\{I, A_t\}$, so $M = aI + bA_t$ for some $a,b \in \mathbb{R}$ (true whether $t=0$ or not). Then

$$MA_t = (aI+bA_t)A_t = aIA_t + bA_tA_t = aA_tI + bA_tA_t = A_t(aI+bA_t) = A_tM$$

Thus $M \in W_t$, so $V_t \subseteq W_t$.

When $t=0$, $\dim V_t = 1$ and $\dim W_t = 4$, so the two are not equal.
When $t \neq 0$, $\dim V_t = 2$ and $\dim W_t = 2$ and $V_t \subseteq W_t$. Thus V_t is a 2-dimensional subspace of the 2-dimensional vector space W_t, so $V_t = W_t$ for $t \neq 0$.