Problem Set #8

(2.5.1) (a) For the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{bmatrix}$, which of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in the margin (if any) are in the kernel of A?

We can just multiply A times each of the vectors $\vec{v}$ to see if we get $\vec{0}$.

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \text{ so } A\vec{v}_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

yes, $\vec{v}_1 \in \ker A$.

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ so } A\vec{v}_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

no, $\vec{v}_2$ is not in $\ker A$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 2 \\ -2 \end{bmatrix}, \text{ so } A\vec{v}_3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

yes, $\vec{v}_3 \in \ker A$

(b) Which vectors have the right height to be in the kernel of T? To be in its image? Can you find a nonzero element in its kernel?

$$T = \begin{bmatrix} 2 & -1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 3 & 0 \\ 2 & -1 & 1 & 0 & 1 \end{bmatrix}, \quad \vec{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{w}_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{w}_4 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

T is a 3×5 matrix, so it is a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^3$. Thus any vector in the kernel must have height 5, and any vector in the image must have height 3 Thus $\vec{w}_4$ has the right height to be in the kernel and $\vec{w}_1$ and $\vec{w}_3$ have the right height to be in the image.

$$\begin{bmatrix} 2 & -1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 3 & 0 \\ 2 & -1 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 2 & -1 & 3 & 2 & 1 \\ 2 & -1 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & -1 & 1 & -4 & 1 \\ 0 & -1 & -1 & -6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & 4 & -1 \\ 0 & 0 & -2 & -2 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & 4 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 5 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

So if $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \in \ker T$, then $a = -2d$, $b = -5d + e$, $c = -d$, so $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -2d \\ -5d+e \\ -d \\ d \\ e \end{bmatrix}$

Pick any values of d and e (not both zero) for an answer. If $d=0$, $e=1$, get $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(2.5.2) Let the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ be onto. Are the following statements true or false?

(a) The columns of T span $\mathbb{R}^n$

False: the columns of T are in $\mathbb{R}^m$, not $\mathbb{R}^n$.

(b) T has rank m

True: by the first grey box on page 201

(c) For any vector $\vec{v} \in \mathbb{R}^m$, there exists a solution to $T\vec{x} = \vec{v}$.

True: by the definition of onto.

(d) For any vector $\vec{v} \in \mathbb{R}^n$, there exists a solution to $T\vec{x} = \vec{v}$.

False: $T: \mathbb{R}^n \to \mathbb{R}^m$, so $\vec{v} \in \mathbb{R}^m$, not $\mathbb{R}^n$.

(e) T has nullity n.

False: If nullity were n, then $\dim \operatorname{Im} T = 0$ by the dimension formula.

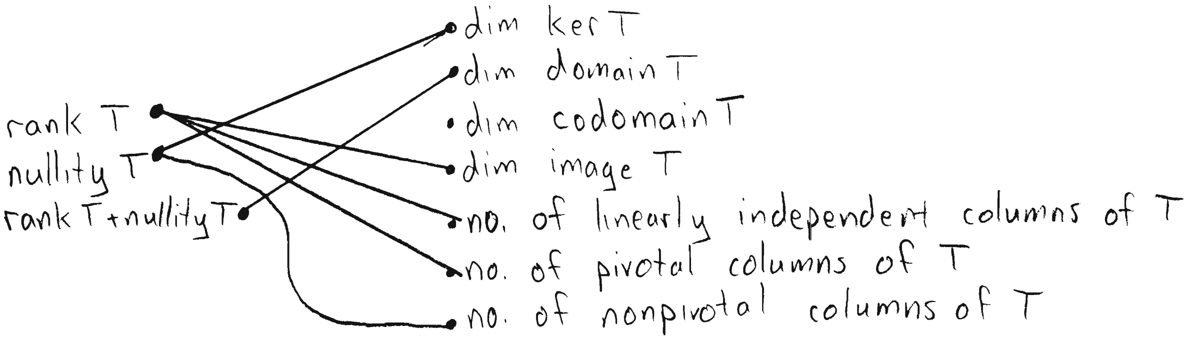
(f) The kernel of T is $\{\vec{0}\}$

False: If $n > m$, then nullity of T is $(n-m) > 0$ by the dimension formula.

(g) For any vector $\vec{v} \in \mathbb{R}^m$, there exists a unique solution to $T\vec{x} = \vec{v}$.

False: If $n > m$, then nullity of T is > 0, so there are columns without pivots, and hence infinitely many solutions to $T\vec{x} = \vec{v}$.

(2.5.3) Let T be a linear transformation Connect each item in the column at left with all synonymous items in the column at right:



All of these follow more or less from the definitions, so I will not explain each pairing separately.

(2.5.5) Prove proposition 2.5.2.

Proposition 2.5.2 states that if $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\ker T$ is a vector subspace of $\mathbb{R}^n$ and $\operatorname{img} T$ is a vector subspace of $\mathbb{R}^m$.

Recall that to prove that something is a subspace, we must demonstrate closure under addition and scalar multiplication We will begin with $\ker T$.

Let $\vec{x}, \vec{y} \in \ker T$, so that $T\vec{x} = \vec{0}$ and $T\vec{y} = \vec{0}$. Let $c \in \mathbb{R}$. Then

$$T(\vec{x}+\vec{y}) = T\vec{x} + T\vec{y} = \vec{0} + \vec{0} = \vec{0} \qquad \text{so } (\vec{x}+\vec{y}) \in \ker T.$$
$$T(c\vec{x}) = cT\vec{x} = c\vec{0} = \vec{0} \qquad \text{so } (c\vec{x}) \in \ker T$$

Thus $\ker T$ is a subspace of $\mathbb{R}^n$.

We proceed similarly for $\operatorname{Img} T$.

Let $\vec{u}, \vec{v} \in \operatorname{Img} T$, so that $\vec{u} = T(\vec{a})$ and $\vec{v} = T(\vec{b})$ for some $\vec{a}, \vec{b} \in \mathbb{R}^n$. Then

$$T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b}) = \vec{u} + \vec{v}, \quad \text{so } (\vec{u}+\vec{v}) \in \operatorname{Img} T.$$
$$T(c\vec{a}) = cT(\vec{a}) = c\vec{u}, \quad \text{so } (c\vec{u}) \in \operatorname{Img} T.$$

Thus $\operatorname{Img} T$ is a subspace of $\mathbb{R}^m$.

Q.5.6) For each of the matrices in the margin, find a basis for the kernel and a basis for the image.

(a) $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ The only pivot is in the first column, so a basis for the image is the first column in the original matrix. That is, basis for image is $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

Now let $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be in the kernel. Then $a+b+3c=0$, so $a=-b-3c$.

Hence $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b-3c \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} -3c \\ 0 \\ c \end{bmatrix} = b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Therefore our basis for the kernel is $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(b) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ -1 & 4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 6 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4/3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 4/3 \\ 0 & 0 & 0 \end{bmatrix}$.

The pivots are in the first and second columns, so the basis for the image is

$$\left\{\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}\right\}$$

For the kernel, let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \ker$, so that $a + \frac{1}{3}c = 0$ and $b + \frac{4}{3}c = 0$.

Then $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}c \\ -\frac{4}{3}c \\ c \end{bmatrix} = c\begin{bmatrix} -1/3 \\ -4/3 \\ 1 \end{bmatrix}$. Hence a basis for the kernel is $\left\{\begin{bmatrix} -1/3 \\ -4/3 \\ 1 \end{bmatrix}\right\}$.

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Then the image has a basis $\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$

For the kernel, let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \ker$, so that $a-c=0$ and $b+2c=0$

Then $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ -2c \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Hence a basis for the kernel is $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$.

(2.5.8) True or false: Let $f: \mathbb{R}^m \to \mathbb{R}^k$ and $g: \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Then $f \circ g = 0$ implies $\operatorname{img} g = \ker f$

This is false. We create a counterexample here (many other counterexamples exist). Let $k=m=n=1$, so that f and g are simply usual functions. Let $f(x)=0$ for all $x \in \mathbb{R}^m$ and $g(y)=0$ for all $y \in \mathbb{R}^n$. These f and g are linear (feel free to check).

$$\mathbb{R}^n \xrightarrow{g=0} \mathbb{R}^m \xrightarrow{f=0} \mathbb{R}^k$$

Then $f(g(y))=0$ for all $y \in \mathbb{R}^n = \mathbb{R}$. However, $\operatorname{img} g = \{\vec{0}\}$, since g maps everything to $\vec{0}=0$, while $\ker f = \mathbb{R}^m = \mathbb{R}$ since f sends all of $\mathbb{R}^m$ to $\vec{0} \in \mathbb{R}^k$. Thus $\operatorname{img} g \neq \ker f$.

(2.5.9) Let P_2 be the space of polynomials of degree ≤ 2, identified with $\mathbb{R}^3$ by identifying $a+bx+cx^2$ to $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

(a) Write the matrix of the linear transformation $T: P_2 \to P_2$ given by

$$(T(p))(x) = xp'(x) + x^2 p''(x)$$

In order to find the matrix of the linear transformation T, we need to find $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$, where $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is the standard basis. Note that

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longleftrightarrow 1 + 0x + 0x^2 = 1$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longleftrightarrow 0 + x + 0x^2 = x$$

$$\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longleftrightarrow 0 + 0x + x^2 = x^2$$

Thus, after our identification, the standard basis of P_2 is $\{1, x, x^2\}$.
For $\vec{e}_1 = p = 1$, observe that $p'(x) = 0$ and $p''(x) = 0$, so $T(1) = x \cdot 0 + x^2 \cdot 0 = 0$.
For $\vec{e}_2 = p = x$, observe that $p'(x) = 1$ and $p''(x) = 0$, so $T(x) = x \cdot 1 + x^2 \cdot 0 = x$.
For $\vec{e}_3 = p = x^2$, observe that $p'(x) = 2x$ and $p''(x) = 2$, so $T(x^2) = x \cdot 2x + x^2 \cdot 2 = 4x^2$

Thus, the matrix of T is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

where first column of the matrix is $T(\vec{e}_1) = 0$, the second column is $T(\vec{e}_2) = x$, and the third column is $T(\vec{e}_3) = 4x^2$.

(b) Find the basis for the image and the kernel of T.

First we row reduce the matrix of T to get $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

Call this matrix A. Let $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b} \in \ker T$, so that $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Then $b_2 = 0$ and $b_3 = 0$, so $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Thus the basis for $\ker T$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(continued) $\rightarrow$

(2.5.9 (b) continued)

Since the second and third columns of A are pivots, our basis for Img T is the second and third columns of the original matrix of T. That is,

$$\text{basis of } \operatorname{Im} T = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$$

(2.5.10) Show that there exist numbers $c_0, \dots, c_k$ such that

$$\int_0^k p(t)\,dt = \sum_{i=0}^{k} c_i p(i) \quad \text{for all polynomials } p \in P_k$$

Let $p \in P_k$ so that $p(t) = a_0 + a_1 t + \cdots + a_k t^k$ for some $a_0, \dots, a_k \in \mathbb{R}$. Let's work with the left hand side first.

$$\int_0^k p(t)\,dt = \int_0^k a_0 + a_1 t + \cdots + a_k t^k\,dt = \left(a_0 t + a_1 \frac{t^2}{2} + \cdots + \frac{a_k t^k}{k+1}\right)\Big|_0^k$$
$$= a_0 k + a_1 \frac{k^2}{2} + \cdots + \frac{a_k k^k}{k+1}$$
$$= \begin{bmatrix} a_0 & \cdots & a_k \end{bmatrix} \begin{bmatrix} k \\ \frac{k^2}{2} \\ \vdots \\ \frac{k^k}{k+1} \end{bmatrix}$$

Now let's work with the right hand side:

$$\sum_{i=0}^{k} c_i p(i) = \begin{bmatrix} p(0) & \cdots & p(k) \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_k \end{bmatrix}$$

Now note that $p(i) = a_0 + a_1 i + \cdots a_k i^k = \begin{bmatrix} a_0 & \cdots & a_k \end{bmatrix} \begin{bmatrix} i^0 \\ i^1 \\ \vdots \\ i^k \end{bmatrix}$

Thus $\begin{bmatrix} p(0) & \cdots & p(k) \end{bmatrix} = \begin{bmatrix} a_0 & \cdots & a_k \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & k \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2^k & \cdots & k^k \end{bmatrix}$

Therefore, we have reduced the question to asking whether there are constants $c_0, \dots, c_k$ such that

$$\begin{bmatrix} a_0 & \cdots & a_k \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & k \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2^k & \cdots & k^k \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} a_0 & \cdots & a_k \end{bmatrix} \begin{bmatrix} k \\ \frac{k^2}{2} \\ \vdots \\ \frac{k^k}{k+1} \end{bmatrix}$$

Then notice that it is sufficient to show that there are $c_0, \dots, c_k$ such that

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & k \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2^k & \cdots & k^k \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} k \\ \frac{k^2}{2} \\ \vdots \\ \frac{k^k}{k+1} \end{bmatrix}$$

for then we can multiply both sides on the left by $\begin{bmatrix} a_0 & \cdots & a_k \end{bmatrix}$. Denote the square matrix on the left by A.

(continued) →

(2.5.10 continued)

If we can show that A is invertible, then the proof will be complete, since then we can solve for the c_i. Observe

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & k \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2^k & \cdots & k^k \end{bmatrix}$$

I will show that the only solution to $A\vec{b} = \vec{0}$ is $\vec{b} = \vec{0}$. Thus the kernel of A will be zero-dimensional, so every column of A will have a pivot, so A will row reduce to the identity, so A will be invertible, and the proof will be complete.

I am going to work with the transpose of A instead of A. This is alright since A and A^T have the same number of pivots for any matrix.

$$A^T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & & 1 \\ 1 & 2 & 2^2 & & 2^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & k & k^2 & & k^k \end{bmatrix}$$

Now observe that the first column of A^T is just the monomial 1 evaluated at $0, \ldots, k$. Similarly, the second column is the monomial t evaluated at $0, \ldots, k$. This continues until the last column is just t^k evaluated at $0, \ldots, k$. Now let

$$\vec{b} = \begin{bmatrix} b_0 \\ \vdots \\ b_k \end{bmatrix}$$

Then requiring $A^T\vec{b} = \vec{0}$ says that $b_0 + b_1 t + \cdots + b_k t^k = 0$ for all of $t = 0, \ldots, k$. But any nonzero polynomial of degree n has at most n zeros (by calculus), so this is impossible unless $b_0 = b_1 = \cdots = b_k = 0$. Hence $A\vec{b} = \vec{0}$ has only one solution, so A^T is injective, and therefore invertible. Thus A is invertible as well, so we can solve for the c_i, and the proof is complete.

(2.5.15) Show that if A and B are $n \times n$ matrices, and AB is invertible, then A and B are invertible.

We know that $(AB)^{-1}$ exists, but we do not know that $(AB)^{-1} = B^{-1}A^{-1}$, since the theorem which gives us that equality (see page 48) <u>assumes</u> that A and B are invertible.

I will first show that A is invertible. Let $C = (AB)^{-1}$, so that $ABC = I$. Then Let $D = BC$, so that $AD = I$. Let $\vec{d}_i$ denote the i^{th} column of D. Then

$$A\vec{d}_i = i^{th} \text{ column of } I$$
$$= \vec{e}_i \quad (\text{the } i^{th} \text{ standard basis vector.}$$

Thus we see that $A: \mathbb{R}^n \to \mathbb{R}^n$ is onto, for if $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, then

$$A(x_1\vec{d}_1 + x_2\vec{d}_2 + \cdots x_n\vec{d}_n) = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$
$$= \vec{x}$$

Thus, by the grey box on page 201, the row reduced matrix $\hat{A}$ has a pivot in every row. Since A is square, this gives a pivot in every column as well, so $\hat{A} = I$, and hence A is invertible.

To prove that B is invertible, let C be as before, so that

$$ABC = I$$
$$A^{-1}ABC = A^{-1}I$$
$$BC = A^{-1}$$
$$BCA = A^{-1}A$$
$$B(CA) = I$$

Thus B has a right inverse, so the proof we used for A works for B as well.

(2.5.18) (a) Prove that given any distinct points $x_0, \dots, x_k \in \mathbb{R}$ and any numbers $c_0, \dots, c_k$, the polynomial

$$p(x) = \sum_{i=0}^{k} c_i \frac{\prod_{i \neq j} (x - x_j)}{\prod_{i \neq j} (x_i - x_j)}$$

is a polynomial of degree at most k such that $p(x_i) = c_i$.

Let's look at the inside of the sum, so at the individual summand

$$c_i \frac{\prod_{i \neq j} (x - x_j)}{\prod_{i \neq j} (x_i - x_j)}$$

Within this summand, i is fixed. It only varies when we take the sum. Note that the denominator is just a product of numbers $x_i - x_j$ with $x_i \neq x_j$, so that the full denominator is just a constant.

For the numerator, we are taking a product of k terms of the form $(x - x_j)$. There are k such terms because j ranges from 0 to k, but we throw out the case where $i = j$. Thus the numerator is a polynomial of degree k. When we then sum over i, we are adding many polynomials of degree k, so the full polynomial $p(x)$ has degree at most k.

Next we want to show that $p(x_i) = c_i$. A word of warning: we might be tempted to say that $p(x_i) = \sum_{i=0}^{k} c_i \frac{\prod_{i \neq j} (x_i - x_j)}{\prod_{i \neq j} (x_i - x_j)} = \sum_{i=0}^{k} c_i$

This is not correct! The notation is deceiving. Inside the sum, i varies term by term. However, when you evaluate at x_i, you are plugging in a specific x_i (that is, i is a certain number). This is most easily observed with an example. Let $k=2$, so that

$$p(x) = \underbrace{c_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}}_{i=0 \text{ term}} + \underbrace{c_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}}_{i=1 \text{ term}} + \underbrace{c_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}_{i=2 \text{ term}},$$

When I find $p(x_0)$, both the $i=1$ and $i=2$ terms vanish since they have $(x - x_0)$ in the numerator. Meanwhile, for the $i=0$ term, all of the parts cancel except for the c_0, so $p(x_0) = c_0$. Similarly for the others.

In general, plugging x_i into a term (summand) of $p(x)$ corresponding to an index other than i will give zero, since that term will have a copy of $(x - x_i)$ in the numerator. This leaves only the i^{th} term, for which the products do cancel, showing that $p(x_i) = c_i$.

(continued) →

(2.5.18 continued)

(b) For $k=1$, $k=2$, and $k=3$, write the matrix of the transformation $\vec{c} \mapsto p$, where $p = a_0 + a_1 x + \cdots + a_k x^k \in P_k$ is identified to the point $\begin{bmatrix} a_0 \\ \vdots \\ a_k \end{bmatrix} \in \mathbb{R}^{k+1}$.

First let $k=1$, so that $p(x) = c_0 \dfrac{(x-x_1)}{(x_0-x_1)} + c_1 \dfrac{(x-x_0)}{(x_1-x_0)} =$

$$= \left(\frac{-c_0 x_1}{(x_0-x_1)} - \frac{c_1 x_0}{(x_1-x_0)} \right) + \left(\frac{c_0}{(x_0-x_1)} + \frac{c_1}{(x_1-x_0)} \right) x$$

To find the matrix of the transformation, we apply the transformation to the standard basis vectors and make these the columns of our matrix. Our standard basis for the $\vec{c}$ vectors (with $k=1$) are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (corresponding to $c_0 = 1$ and $c_1 = 0$) and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (corresponding to $c_0 = 0$ and $c_1 = 1$). I'll call this linear transformation T. Then

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{-x_1}{(x_0-x_1)} + \frac{1}{(x_0-x_1)} x = \begin{bmatrix} \dfrac{-x_1}{(x_0-x_1)} \\[2ex] \dfrac{1}{(x_0-x_1)} \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{-x_0}{(x_1-x_0)} + \frac{1}{(x_1-x_0)} x = \begin{bmatrix} \dfrac{-x_0}{x_1-x_0} \\[2ex] \dfrac{1}{(x_1-x_0)} \end{bmatrix}$$

Thus the matrix of T is
$$\begin{bmatrix} \dfrac{-x_1}{(x_0-x_1)} & \dfrac{-x_0}{(x_1-x_0)} \\[2ex] \dfrac{1}{(x_0-x_1)} & \dfrac{1}{(x_1-x_0)} \end{bmatrix}$$

For $k=2$, $p(x) = c_0 \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + c_1 \dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + c_2 \dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

Skipping the intermediate steps (very similar to the $k=1$ case) we get the matrix

$$\begin{bmatrix} \dfrac{x_1 x_2}{(x_0-x_1)(x_0-x_2)} & \dfrac{x_0 x_2}{(x_1-x_0)(x_1-x_2)} & \dfrac{x_0 x_1}{(x_2-x_0)(x_2-x_1)} \\[2ex] \dfrac{-x_1-x_2}{(x_0-x_1)(x_0-x_2)} & \dfrac{-x_0-x_2}{(x_1-x_0)(x_1-x_2)} & \dfrac{-x_0-x_1}{(x_2-x_0)(x_2-x_1)} \\[2ex] \dfrac{1}{(x_0-x_1)(x_0-x_2)} & \dfrac{1}{(x_1-x_0)(x_1-x_2)} & \dfrac{1}{(x_2-x_0)(x_2-x_1)} \end{bmatrix}$$

The $k=3$ case is a rather complicated 4×4 matrix. It isn't too hard if you see the pattern, but I think we get how this works by now, so I told the students they could skip this case.