Problem Set #3

(1.7.2) For what a is the tangent to the graph of $f(x)=e^{-x}$ at $\begin{pmatrix} a \\ e^{-a} \end{pmatrix}$ a line of the form $y=mx$?

We can compute the slope in two different ways. First, the slope of the tangent line is the value of the derivative $f'(a) = -e^{-a}$. Second, if this line is of the form $y=mx$, then it also passes through the origin, so the slope is e^{-a}/a by rise over run. Thus

$$-e^{-a} = m = \frac{e^{-a}}{a}$$

Dividing by e^{-a}, we get $a=-1$ is the only solution.

(1.7.4) Example 1.7.2 may lead you to expect that if f is differentiable at a, then $f(a+h) - f(a) - f'(a)h$ has something to do with h^2. It is not true that once you get rid of the linear term you always have a term that involves h^2. Using the definition, compute the derivative at 0 of

(a) $f(x) = |x|^{3/2}$

$$\lim_{h \to 0} \frac{1}{h}\left(f(0+h) - f(0)\right) = \lim_{h\to 0} \frac{1}{h}\left(|h|^{3/2}\right)$$

Now if h is positive, then $|h|^{3/2} = h^{3/2}$, so $\lim_{h\to 0^+} \frac{1}{h}\left(|h|^{3/2}\right) = \lim_{h\to 0^+} h^{1/2} = 0$

If h is negative, then $\lim_{h\to 0^-} \frac{1}{h}\left(|h|^{3/2}\right) = \lim_{h\to 0^-} -\frac{1}{|h|}\left(|h|^{3/2}\right) = \lim_{h\to 0^-} -|h|^{1/2} = 0$

Thus the limit exists, so $f(x)$ is differentiable at 0 with derivative 0.

(b) $f(x) = x \ln|x|$

(c) $f(x) = \dfrac{x}{\ln|x|}$

Neither ~~of~~ the functions in (b) or (c) are even <u>defined</u> at $x=0$, so the definition of derivative is not satisfied. There are different ways to try to make sense out of this using limits, but I talked it over with Professor Meyer, and we decided to just cancel these parts instead.

(1.7.5) What are the partial derivatives $D_1 f$ and $D_2 f$ of the functions in the margin, at the points $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-2\end{pmatrix}$?

(a) $f\begin{pmatrix}x\\y\end{pmatrix} = \sqrt{x^2+y}$

$D_1 f(x,y) = \frac{1}{2}2x(x^2+y)^{-1/2}$
$D_1 f(2,1) = 2(4+1)^{-1/2}$
$= 2/\sqrt{5}$
$D_1 f(1,-2) =$ not defined

$D_2 f(x,y) = \frac{1}{2}(x^2+y)^{-1/2}$
$D_1 f(2,1) = 1/2\sqrt{5}$
$D_2 f(1,-2) =$ not defined

(b) $f\begin{pmatrix}x\\y\end{pmatrix} = x^2y + y^4$

$D_1 f\begin{pmatrix}x\\y\end{pmatrix} = 2xy$
$D_1 f\begin{pmatrix}2\\1\end{pmatrix} = 4$
$D_1 f\begin{pmatrix}1\\-2\end{pmatrix} = -4$

$D_2 f\begin{pmatrix}x\\y\end{pmatrix} = x^2 + 4y^3$
$D_2 f\begin{pmatrix}2\\1\end{pmatrix} = 8$
$D_2 f\begin{pmatrix}1\\-2\end{pmatrix} = -31$

(c) $f\begin{pmatrix}x\\y\end{pmatrix} = \cos(xy) + y\cos(y)$

$D_1 f\begin{pmatrix}x\\y\end{pmatrix} = -y\sin(xy)$
$D_1 f\begin{pmatrix}2\\1\end{pmatrix} = -\sin(2)$
$D_1 f\begin{pmatrix}1\\-2\end{pmatrix} = 2\sin(-2)$

$D_2 f\begin{pmatrix}x\\y\end{pmatrix} = -x\sin(xy) - y\sin(y) + \cos(y)$
$D_2 f\begin{pmatrix}2\\1\end{pmatrix} = -2\sin(2) - \sin(1) + \cos(1)$
$D_2 f\begin{pmatrix}1\\-2\end{pmatrix} = -\sin(-2) + 2\sin(-2) + \cos(-2)$

(d) $f\begin{pmatrix}x\\y\end{pmatrix} = \dfrac{xy^2}{\sqrt{x+y^2}}$

$$D_1 f\begin{pmatrix}x\\y\end{pmatrix} = \frac{y^2\sqrt{x+y^2} - xy^2\cdot\frac{1}{2}(x+y^2)^{-1/2}}{x+y^2}$$
$$D_1 f\begin{pmatrix}2\\1\end{pmatrix} = \frac{\sqrt{3} - (3)^{-1/2}}{3} = \frac{3-1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}$$
$$D_1 f\begin{pmatrix}1\\-2\end{pmatrix} = \frac{4\sqrt{5} - 2(5)^{-1/2}}{5} = \frac{20-2}{5\sqrt{5}} = \frac{18}{5\sqrt{5}}$$

$$D_2 f\begin{pmatrix}x\\y\end{pmatrix} = \frac{2xy\sqrt{x+y^2} - xy^2\cdot\frac{1}{2}\cdot 2y(x+y^2)^{-1/2}}{x+y^2}$$
$$D_2 f\begin{pmatrix}2\\1\end{pmatrix} = \frac{4\sqrt{3} - 2(3)^{-1/2}}{3} = \frac{12-2}{3\sqrt{3}} = \frac{10}{3\sqrt{3}}$$
$$D_2 f\begin{pmatrix}1\\-2\end{pmatrix} = \frac{-4\sqrt{5} + 8(5)^{-1/2}}{5} = \frac{-20+8}{5\sqrt{5}} = \frac{-12}{5\sqrt{5}}$$

(1.7.6) Calculate the partial derivatives $\frac{\partial \bar{f}}{\partial x}$ and $\frac{\partial \bar{f}}{\partial y}$ for the $\mathbb{R}^m$ valued functions

(a) $\bar{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos x \\ x^2 y + y^2 \\ \sin(x^2 - y) \end{pmatrix}$ $\quad \frac{\partial \bar{f}}{\partial x} = \begin{pmatrix} -\sin x \\ 2xy \\ 2x\cos(x^2-y) \end{pmatrix}$ $\quad \frac{\partial \bar{f}}{\partial y} = \begin{pmatrix} 0 \\ x^2 + 2y \\ -\cos(x^2-y) \end{pmatrix}$

(b) $\bar{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ xy \\ \sin^2 xy \end{pmatrix}$ $\quad \frac{\partial \bar{f}}{\partial x} = \begin{pmatrix} x(x^2+y^2)^{-1/2} \\ y \\ 2y\sin(xy)\cos(xy) \end{pmatrix}$ $\quad \frac{\partial \bar{f}}{\partial y} = \begin{pmatrix} y(x^2+y^2)^{-1/2} \\ x \\ 2x\sin(xy)\cos(xy) \end{pmatrix}$

(1.7.7) Write the answers to exercise 1.7.6 in the form of the Jacobian matrix.

(a) $J\bar{f}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} -\sin x & 0 \\ 2xy & x^2+2y \\ 2x\cos(x^2-y) & -\cos(x^2-y) \end{pmatrix}$

(b) $J\bar{f}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} x(x^2+y^2)^{-1/2} & y(x^2+y^2)^{-1/2} \\ y & x \\ 2y\sin(xy)\cos(xy) & 2x\sin(xy)\cos(xy) \end{pmatrix}$

(1.7.10) Let $\vec{F}:\mathbb{R}^2 \to \mathbb{R}^2$ be a function.

(a) Prove that if $\vec{F}$ is linear, then for any $\vec{a},\vec{v}\in\mathbb{R}^2$,

$$\vec{F}\begin{pmatrix} a_1+v_1 \\ a_2+v_2 \end{pmatrix} = \vec{F}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + [D\vec{F}(\vec{a})]\vec{v} \qquad (*)$$

To say that $\vec{F}:\mathbb{R}^2\to\mathbb{R}^2$ is linear means that

$$\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

where each of f_1, and f_2 are linear, i.e.

$$f_1(x,y) = b_1 + b_2x + b_3y$$
$$f_2(x,y) = c_1 + c_2x + c_3y$$

Then $[D\vec{F}(\vec{a})] = \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}$

Hence the left hand side of $(*)$ is equal to

$$\vec{F}\begin{pmatrix} a_1+v_1 \\ a_2+v_2 \end{pmatrix} = \begin{bmatrix} b_1 + b_2(a_1+v_1) + b_3(a_2+v_2) \\ c_1 + c_2(a_1+v_1) + c_3(a_2+v_2) \end{bmatrix}$$

And the right hand side is

$$\vec{F}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + [D\vec{F}(\vec{a})]\vec{v} = \begin{bmatrix} b_1 + b_2a_1 + b_3a_2 \\ c_1 + c_2a_1 + c_3a_2 \end{bmatrix} + \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$= \begin{bmatrix} b_1 + b_2a_1 + b_3a_2 \\ c_1 + c_2a_1 + c_3a_2 \end{bmatrix} + \begin{bmatrix} b_2v_1 + b_3v_2 \\ c_2v_1 + c_3v_2 \end{bmatrix}$$
$$= \text{right hand side.}$$

(b) Prove that if $\vec{F}$ is not linear, then this is not true.

The easiest way to prove this is just to point out that the left hand side of $(*)$ is the value of $\vec{F}$ at $(\vec{a}+\vec{v})$, while the right hand side is the value of the tangent plane, based at $\vec{a}$, and evaluated at $(\vec{a}+\vec{v})$. Thus, unless $\vec{F}$ is equal to it's tangent plane (that is, unless $\vec{F}$ is a plane), then there is some $\vec{v}$ so that

$$\vec{F}(\vec{a}+\vec{v}) \neq \vec{F}(\vec{a}) + [D\vec{F}(\vec{a})]\vec{v}$$

So if $\vec{F}$ is nonlinear, then at least one of f_1, f_2 is not a plane. Hence, the equality does not hold for all $\vec{a},\vec{v}\in\mathbb{R}^2$.

(1.7.11) Find the Jacobian matrices of the following mappings.

(a) $f\begin{pmatrix} x \\ y \end{pmatrix} = \sin(xy)$ $\qquad J f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} y\cos xy & x\cos xy \end{bmatrix}$

(b) $f\begin{pmatrix} x \\ y \end{pmatrix} = e^{(x^2+y^3)}$ $\qquad J f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2xe^{(x^2+y^3)} & 3y^2 e^{(x^2+y^3)} \end{bmatrix}$

(c) $\bar{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix}$ $\qquad J\bar{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$

(d) $\bar{f}\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$ $\qquad J\bar{f}\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$

(1.7.13) Show that if $f(x) = |x|$, then for any number m,
$$\lim_{h\to 0} (f(0+h) - f(0) - mh) = 0$$

but that $\lim_{h\to 0} \frac{1}{h}(f(0+h) - f(0) - mh) = 0$ is never true: there is no number m such that mh is a "good approximation" to $f(h) - f(0)$ in the sense of definition 1.7.5.

Let's prove the first limit:
$$\lim_{h\to 0} (f(0+h) - f(0) - mh) = \lim_{h\to 0} (|h| - 0 - mh) = 0$$, as claimed.

For the second limit, we split into two cases, when h approaches 0 from the right (so from positive side), and when h approaches 0 from the (negative) left.
$$\lim_{h\to 0^+} \frac{1}{h}(f(0+h) - f(0) - mh) = \lim_{h\to 0^+} \frac{1}{h}(|h| - mh) = \lim_{h\to 0} \frac{1}{h}((1-m)h) = 1-m$$
$$\lim_{h\to 0^-} \frac{1}{h}(f(0+h) - f(0) - mh) = \lim_{h\to 0} \frac{1}{h}(|h| - mh) = \lim_{h\to 0} \frac{1}{h}((-1-m)h) = -1-m$$

So in order for both of these limits to be zero, m must simultaneously be equal to 1 and -1, an impossibility.