Problem Set #6

(3.1.2) Show that the set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x + x^2 + y^2 = 2 \right\}$ is a smooth curve.

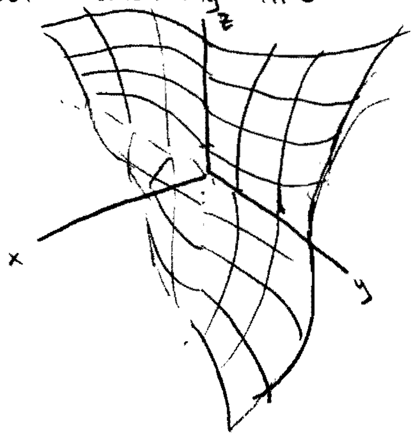
Let $F\begin{pmatrix} x \\ y \end{pmatrix} = x + x^2 + y^2 - 2$, so that our set is given by $F\begin{pmatrix} x \\ y \end{pmatrix} = 0$. The by Theorem 3.1.10, we just need to check that $\left[DF\begin{pmatrix} x \\ y \end{pmatrix}\right]$ is onto for all points in our set.

We calculate $\left[DF\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{bmatrix} 1+2x & 2y \end{bmatrix}$.

This matrix would not be only only if $x = -\frac{1}{2}$ and $y = 0$, but $\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$ is not a point of our set, so the given set is indeed a smooth curve.

(3.1.6) What does the surface of equation $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sin(x+yz) = 0$ in example 3.1.13 look like?

Note that $\sin(x+yz)=0$ iff $x+yz = k\pi$ for some $k \in \mathbb{Z}$. If $k=0$ (for example) then $x=-yz$ looks something like



(in case you can't tell, I have no drawing talent)

Anyway, the important issue is that we will have one of these surfaces for each value of k, offset by π in the x-direction. Thus the full manifold will have infinitely disjoint sheets.

(3.1.7)(a) Show that for all a and b, the sets X_a and Y_b of equations

$$x^2+y^3+z=a \qquad \text{and} \qquad x+y+z=b,$$

respectively, are smooth surfaces.

In both equations, we can solve for z as a function of x and y everywhere, so X_a and Y_b are smooth surfaces.

(b) For what values of a and b does Theorem 3.1.10 guarantee that the intersection $X_a \cap Y_b$ is a smooth curve? What geometric relation is there between the surfaces X_a and Y_b for the other values of a and b?

Let $F: \mathbb{R}^3 \to \mathbb{R}^2$ be given by $\begin{pmatrix} x^2+y^3+z-a \\ x+y+z-b \end{pmatrix}$

so that $X_a \cap Y_b$ = the set on which F is zero. Then calculate

$$\left[DF\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{bmatrix} 2x & 3y^2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

This matrix is onto unless all columns are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, that is, unless $x=1/2$, $y=\pm 1/\sqrt{3}$. Now we just need to figure out which values of a and b allow these to lie in $X_a \cap Y_b$. So

$$1/4 \pm (3)^{-3/2} + z = a$$
$$1/2 \pm (3)^{-1/2} + z = b$$

Subtracting these equations, we get

$$-1/4 \pm \frac{1}{3\sqrt{3}} \mp \frac{1}{\sqrt{3}} = a-b$$

$$\text{or} \qquad -1/4 \pm \frac{1}{2\sqrt{3}} = a-b$$

This corresponds to a '1-parameter' subset of the possible (a,b) on which $X_a \cap Y_b$ is not a smooth curve.

At the points where $X_a \cap Y_b$ is not a smooth curve, $\left[DF\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right]$ is not onto, so

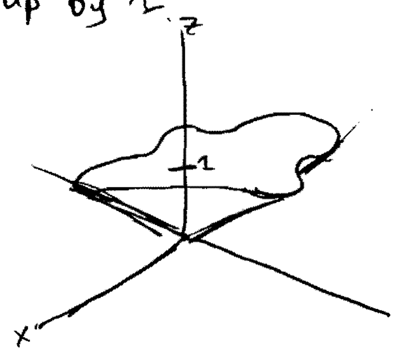
$$\left[DF\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Note that $x+y+z=b$ is a plane, so this condition on the derivative tells us that our two surfaces have the same tangent plane.

(3.1.10) Let $f\begin{pmatrix} x \\ y \end{pmatrix}=0$ be the equation of a curve $X \subset \mathbb{R}^2$, and suppose that $\left[Df\begin{pmatrix} x \\ y \end{pmatrix}\right] \neq 0$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \in X$.

(a) Find an equation for the closure $\overline{CX}$ of the cone $CX \subset \mathbb{R}^3$ over X, where CX is the union of all the lines through the origin and points $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ with $\begin{pmatrix} x \\ y \end{pmatrix} \in X$.

Let $F(x,y,z)=0$ be the equation for the closure that we are looking for. If $(x,y) \in X$, then $f(x,y)=0$, so $F(x,y,1)=0$. Thus the slice of CX at height 1 just looks like the curve X shifted up by 1.



Then we have lines from each of these points to the origin, so if $F(x,y,1)=0$, then $F(xz, yz, z)=0$ as well. This is because any multiple of $(x,y,1)$ lies on the line from $(x,y,1)$ to the origin, so it lies in CX. So we need an equation so that if we multiply all of x,y,z by a constant, we'll still get 0. Use

$$F(x,y,z) = f\left(\frac{x}{z}, \frac{y}{z}\right).$$

(b) If X has the equation $y=x^3$, what is the equation of $\overline{CX}$?

$$f(x,y) = y - x^3 = 0, \quad \text{so} \quad F(x,y,z) = f\left(\frac{x}{z}, \frac{y}{z}\right) = \frac{y}{z} - \frac{x^3}{z^3} = 0$$

This becomes $z^2 y = x^3$ if we clear the denominators.

(c) When X has the equation $y=x^3$, where is $\overline{CX} - \{0\}$ guaranteed to be a smooth surface?

The equation is $F(x,y,z) = z^2y - x^3$. If we apply the implicit function theorem, we note that F is C^1, and has Jacobian

$$[DF(x,y,z)] = \begin{bmatrix} -3x^2 & z^2 & 2zy \end{bmatrix}$$

This could fail to be onto only if $x=z=0$, so $\overline{CX} - \{0\}$ is a smooth surface except possibly on the y-axis.

(3.1.11) (a) Find a parametrization for the union X of the lines through the origin and a point of the parametrized curve $t \longmapsto \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$.

Given a certain point $\begin{pmatrix} a \\ a^2 \\ a^3 \end{pmatrix}$, the line through this and the origin is the set of points $\begin{pmatrix} sa \\ sa^2 \\ sa^3 \end{pmatrix}$ where $s \in \mathbb{R}$. Thus X is parametrized by

$$\begin{pmatrix} s \\ t \end{pmatrix} \longmapsto \begin{pmatrix} st \\ st^2 \\ st^3 \end{pmatrix}$$

(b) Find an equation for the closure $\overline{X}$ of X. Is $\overline{X}$ exactly X?

Since X is two dimensional, this is just asking us to find a single equation which $x=st$, $y=st^2$, and $z=st^3$ satisfy, with no parameters s,t remaining. Such an equation is

$$xz=y^2$$

The zero set of $xz-y^2=0$ is certainly closed, but it contains more than X. Specifically, this equation contains the x and z axes. While these are not in X, the x-axis is the limit of lines in X as $t \to 0$, and the z-axis is the limit as $t \to \infty$. Thus the zero set of $xz-y^2$ is contained in the closure of X, and is closed, so $\overline{X}$ is equal to the zero set of $xz-y^2$.

(c) Show that $\overline{X} - \{\vec{0}\}$ is a smooth surface.

We apply the implicit function theorem and note that the Jacobian for $f = xz-y^2$ is equal to

$$[Df(x,y,z)] = [z \quad -2y \quad x]$$

This fails to be onto only at the origin, which we have excluded. Also, f is C^1, so by the implicit function theorem, $\overline{X} - \{\vec{0}\}$ is a smooth surface.

(d) Show that the map $\begin{pmatrix} r \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} r(1+\sin\theta) \\ r\cos\theta \\ r(1-\sin\theta) \end{pmatrix}$ is another parametrization of $\overline{X}$.

In this form you should have no trouble giving a name to the surface $\overline{X}$.

To show this is another parametrization, just plug it in to the equation $xz-y^2=0$. So

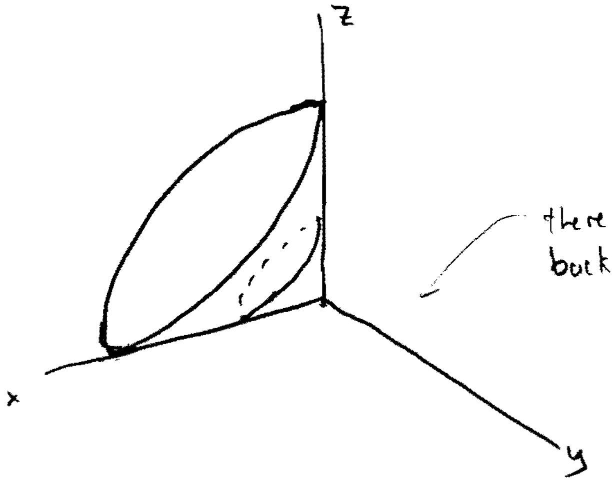
$$r(1+\sin\theta)r(1-\sin\theta) - r^2\cos^2\theta = r^2(1-\sin^2\theta-\cos^2\theta)$$
$$= 0$$

Thus this is a parametrization of $\overline{X}$.

(continued)

(3.1.11(d) continued)

To figure out what this shape is, I suggest fixing r and letting θ vary. Then these will look like circles brushing the x and z-axes, so as r varies, we get a double cone based at the origin, and tilted so that it's axis lies in the xz-plane along $x=z$.



there is anothe cone coming out the back, but not drawn in

(e) Relate $\bar{X}$ to the set of of noninvertible symmetric 2×2 matrices.

A symmetric matrix is of the form $\begin{bmatrix} a & b \\ b & d \end{bmatrix} = A$

and if A is noninvertible, then $ad-b^2=0$. This is the same equation that our closure $\bar{X}$ satisfies.

(3.1.13) (a) What is the equation of the plane $P \subset \mathbb{R}^3$ that contains the point $\bar{a} = (a_1, a_2, a_3)$ and is perpendicular to the vector $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

The equation is $v_1(x-a_1) + v_2(y-a_2) + v_3(z-a_3) = 0$, also written $(\bar{x}-\bar{a})\cdot\bar{v} = 0$

(b) Let $\gamma(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$, and let P_t be the plane through the point $\gamma(t)$ and perpendicular to $\bar{\gamma}'(t)$. What is the equation of P_t?

The equation of P_t is

$$\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}\right] \cdot \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} = 0$$

or equivalently

$$(x-t) + 2t(y-t^2) + 3t^2(z-t^3) = 0$$

(c) Show that if $t_1 \neq t_2$, the planes P_{t_1} and P_{t_2} always intersect in a line. What are the equations of the line $P_1 \cap P_t$?

Two planes in $\mathbb{R}^3$ fail to intersect in a line precisely when they have proportional/ normal (perpendicular) vectors. But if $t_1 \neq t_2$, then

$$\begin{pmatrix} 1 \\ 2t_1 \\ 3t_2 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 2t_2 \\ 3t_3 \end{pmatrix} \cdot a \quad \text{for any constant } a.$$

Next, note that P_1 has equation

$$(x-1) + 2(y-1) + 3(z-1) = 0 \iff x + 2y + 3z - 6 = 0$$

and P_t has equation $\quad x + 2ty + 3t^2 z - t - 2t^3 - 3t^5 = 0$

Thus the intersection is given by both of these equations.

(d) What is the limiting position of the line $P_1 \cap P_{1+h}$ as h tends to 0?

Write these equations as the matrix (of linear equations)

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 2(1+h) & 3(1+h)^2 & (1+h) + 2(1+h)^3 + 3(1+h)^5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 2h & 6h+3h^2 & 22h + 36h^2 + 32h^3 + 15h^4 + 3h^5 \end{bmatrix}$$

continued →

(3.1.13 (d) continued)

$$\sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 2 & 6+3h & 22+36h+32h^2+15h^3+3h^4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3-3h & -16-36h-32h^2-15h^3-3h^4 \\ 0 & 2 & 6+3h & 22+36h+32h^2+15h^3+3h^4 \end{bmatrix}$$

Taking the limit as $h \to 0$, we get

$$\begin{bmatrix} 1 & 0 & -3 & -16 \\ 0 & 1 & 3 & 11 \end{bmatrix}$$

or equivalently,

$$\boxed{\begin{aligned} x - 3z &= -16 \\ y + 3z &= 11 \end{aligned}}\ .$$

(3.1.16) Consider the space X_ℓ of positions of a rod of length ℓ in $\mathbb{R}^3$, whee one endpoint is constrained to be on the x-axis, and the other is constrained to be on the unit sphere centered at the origin.

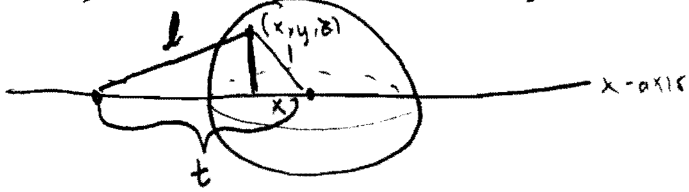
(a) Give equations for X_ℓ as a subset of $\mathbb{R}^4$, where the coordinates in $\mathbb{R}^4$ are the x-coordinate of the end of the rod on the x-axis (call it t), and the other three coordinates of the other end of the rod.

The equations of the surface are

$$x^2+y^2+z^2=1$$

$$\text{and} \quad (t-x)^2 = \ell^2 - y^2 - z^2 \quad \leftrightarrow \quad t^2-2tx+x^2+y^2+z^2-\ell^2=0$$

The second one is slightly harder to see, but arise by looking at the right triangle in



(b) Show that near the point $\bar{p} = \begin{pmatrix} 1+\ell \\ 1 \\ 0 \\ 0 \end{pmatrix}$, the set X_ℓ is a manifold.

We compute $[DF(t,x,y,z)] = \begin{bmatrix} 0 & 2x & 2y & 2z \\ 2t-2x & 2x-2t & 2y & 2z \end{bmatrix}$

If we plug in $\bar{p}$, we get

$$[DF(1+\ell,1,0,0)] = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2\ell & -2\ell & 0 & 0 \end{bmatrix}$$

This is onto for $\ell \neq 0$, so X_ℓ is a manifold in a neighborhood of $\bar{p}$.

(c) Show that for $\ell \neq 1$, X_ℓ is a manifold.

X_ℓ will not be a manifold iff $[DF(t,x,y,z)] = \begin{bmatrix} 0 & 2x & 2y & 2z \\ 2t-2x & 2x-2t & 2y & 2z \end{bmatrix}$

$$\sim \begin{bmatrix} 0 & 2x & 2y & 2z \\ 2t-2x & -2t & 0 & 0 \end{bmatrix}$$

Not all of x,y,z can be zero, so as long as $t \neq x$, this is onto. If $t=x$, then (since $\ell \neq 1$) it is true that $t=0$, $x=0$. Also, (since $\ell \neq 0$), one of y or z is nonzero. Hence

$$\text{nonzero} \rightarrow \begin{bmatrix} 2x \\ -2t \end{bmatrix}, \begin{bmatrix} 2y \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 2z \\ 0 \end{bmatrix} \leftarrow \text{nonzero}$$

are linearly independent, hence onto, so X_ℓ is a manifold.

(3.46 continued)

(d) Show that if $l=1$, there are points of X_1 near which X_1 is not a manifold.

If $l=1$, then consider points of the sort $(0,0,y,z)$, so that one end of the rod is at the origin, and the other end of the rod lies on the circle lying in the sphere with $x=0$.

Then $[DF(0,0,y,z)] = \begin{bmatrix} 0 & 0 & 2y & 2z \\ 0 & 0 & 2y & 2z \end{bmatrix}$

This is not onto, so X_1 is not a manifold near these points

(3.2.1) Exercise 3.1.2 asked you to show that $x + x^2 + y^2 = 2$ defines a smooth curve.

(a) What is an equation for the tangent line to this curve at $\begin{pmatrix}1\\0\end{pmatrix}$?

Note that if we rewrite $F(x,y) := x + x^2 + y^2 - 2$, so that our curve is $F = 0$, then

$$[DF(x,y)] = [1+2x \quad 2y]$$

Hence at $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$,

$$\left[DF\begin{pmatrix}1\\0\end{pmatrix}\right] = [3 \quad 0]$$

Then by Theorem 3.2.4, the tangent space to our curve at $\begin{pmatrix}1\\0\end{pmatrix}$ is $\ker\left[DF\begin{pmatrix}1\\0\end{pmatrix}\right]$.

$$[3 \quad 0]\begin{bmatrix}a\\b\end{bmatrix} = 0$$

Then $a = 0$ and b is arbitrary, so we can describe this by

$$3(x-1) = 0, \text{ or equivalently, } x = 1.$$

(b) What is the equation for the tangent space to the curve at the same point?

The tangent space is $\ker\left[DF\begin{pmatrix}1\\0\end{pmatrix}\right] = \left\{\begin{pmatrix}a\\b\end{pmatrix} \mid a = 0\right\}$, that is, the y-axis, or more precisely, all vectors based at the origin lying on the y-axis.

2.25) (a) Write the equations of the tangent planes P_1, P_2, P_3 to the surface of equation $z = Ax^2 + By^2$ at the points

$$p_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad p_2 = \begin{pmatrix} a \\ 0 \\ Aa^2 \end{pmatrix} \qquad p_3 = \begin{pmatrix} 0 \\ b \\ Bb^2 \end{pmatrix}$$

and find the point $q \in P^1 \cap P^2 \cap P^3$.

Let $F(x,y,z) = Ax^2 + By^2 - z$ so that

$$[DF(x,y,z)] = [2Ax \quad 2By \quad -1]$$

then at our three points,

$$[DF(0,0,0)] = [0 \quad 0 \quad 1]$$
$$[DF(a,0,Aa^2)] = [2Aa \quad 0 \quad -1]$$
$$[DF(0,b,Bb^2)] = [0 \quad 2Bb \quad -1]$$

Thus the equations of our tangent planes are

$$-1(z-0) = 0 \text{ or } z = 0 \quad \text{at } p_1$$
$$2Aa(x-a) - (z - Aa^2) = 0 \quad \text{at } p_2$$
$$2Bb(y-b) - (z - Bb^2) = 0 \quad \text{at } p_3.$$

To find q, we solve these three equations simultaneously, so we get

$$q = \begin{pmatrix} A/2 \\ B/2 \\ 0 \end{pmatrix}$$

(b) What is the volume of the tetrahedron with vertices at p_1, p_2, p_3, and q?

The volume of the parallelopiped with vertices p_2, p_3, and q adjacent to the origin is

$$\det \begin{pmatrix} a & 0 & A/2 \\ 0 & b & B/2 \\ Aa^2 & Bb^2 & 0 \end{pmatrix}$$

so the volume of the tetrahedron is $1/6$ of this, specifically,

$$\frac{1}{6} \begin{vmatrix} a & 0 & A/2 \\ 0 & b & B/2 \\ Aa^2 & Bb^2 & 0 \end{vmatrix} = \frac{1}{6} \left| -\frac{B^2b^2a}{2} - \frac{A^2a^2b}{2} \right|$$

(3.2.8) Give the equation of the tangent space to the set X_ℓ of exercise 3.1.16, at the point $\bar{p} = \begin{pmatrix} 1+\ell \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

We recall that by Theorem 3.2.4, the tangent space $T_{\bar{p}}M$ at $\bar{p}$ is equal to the kernel of $[DF(\bar{p})]$. Recall that (by exercise 3.1.16)

$$[DF(\bar{p})] = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2\ell & -2\ell & 0 & 0 \end{bmatrix}$$

Row reducing to find the kernel, we get (since $\ell \neq 0$)

$$\rightsquigarrow \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The kernel of this matrix is all vectors of the form

$$\begin{bmatrix} 0 \\ 0 \\ a \\ b \end{bmatrix} \qquad a, b \in \mathbb{Z}$$

Equivalently, the tangent space is generated by

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(3.2.9) For the cone CX of exercise 3.1.10, what is the equation of the tangent plane to CX at any point $\overline{x} \in CX - \{\overline{0}\}$.

Note that in CX (not $\overline{CX}$), the only point with z-coordinate 0 is the origin, so we may assume $\overline{x} = (x_0, y_0, z_0)$ has nonzero z-coordinate. Then near $\overline{x}$, the equation of CX is given by $f(x/z, y/z) = 0$, where $f(x,y) = 0$ is the equation of the curve X in $\mathbb{R}^2$ which we used to define the cone.

Then the tangent plane is given by the equation

$$\left[DCX(\overline{x})\right]\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

Here

$$[DCX(\overline{x})] = \left[\frac{1}{z_0} D_1 f\left(\frac{x_0}{z_0}, \frac{y_0}{z_0}\right) \quad \frac{1}{z_0} D_2 f\left(\frac{x_0}{z_0}, \frac{y_0}{z_0}\right) \quad -\frac{x_0}{z_0^2} D_1 f\left(\frac{x_0}{z_0}, \frac{y_0}{z_0}\right) - \frac{y_0}{z_0^2} D_2 f\left(\frac{x_0}{z_0}, \frac{y_0}{z_0}\right) \right]$$

Thus the equation of the tangent plane is

$$\frac{x - x_0}{z_0} D_1 f(x_0/z_0, y_0/z_0) + \frac{y - y_0}{z_0} D_2 f(x_0/z_0, y_0/z_0) - \frac{x_0(z - z_0)}{z_0^2} D_1 f\left(\frac{x_0}{z_0}, y_0/z_0\right) - \frac{y_0(z - z_0)}{z_0^2} D_2 f\left(\frac{x_0}{z_0}, y_0/z_0\right) = 0$$