Problem Set #7

(3.3.2) (a) Write out the polynomial $\sum_{m=0}^{5} \sum_{I \in \mathcal{I}_3^m} a_I \bar{x}^I$, where

$a_{(0,0,0)} = 4$ $\quad a_{(0,1,0)} = 3,$ $\quad a_{(1,0,2)} = 4$ $\quad a_{(1,1,2)} = 2$

$a_{(2,2,0)} = 1$ $\quad a_{(3,0,2)} = 2$ $\quad a_{(5,0,0)} = 3$

and all other $a_I = 0$ for $I \in \mathcal{I}_3^m$ for $m \le 5$.

We get $4 + 3x_2 + 4x_1x_3^2 + 2x_1x_2x_3^2 + x_1^2x_2^2 + 2x_1^3x_3^2 + 3x_1^5$.
Here I used $\bar{x} = (x_1, x_2, x_3)$. Of course you could label your variables $\bar{x} = (x, y, z)$, or something else.

(b) Use multi-exponent notation to write the polynomial

$$2x_2 + x_1x_2 - x_1x_2x_3 + x_1^2 + 5x_2^2x_3$$

We could write this as $\sum_{m=0}^{3} \sum_{I \in \mathcal{I}_3^m} a_I \bar{x}^I$, where the only nonzero a_I are

$a_{(0,1,0)} = 2,$ $\quad a_{(1,1,0)} = 1,$ $\quad a_{(1,1,1)} = -1$

$a_{(2,0,0)} = 1,$ $\quad a_{(0,2,1)} = 5$

(c) Use multi-exponent notation to write the polynomial

$$3x_1x_2 - x_2x_3x_4 + 2x_2^2x_3 + x_2^2x_4^4 + x_2^5$$

We could write this as $\sum_{m=0}^{6} \sum_{I \in \mathcal{I}_4^m} a_I \bar{x}^I$, where the only nonzero a_I are

$a_{(1,1,0,0)} = 3,$ $\quad a_{(0,1,1,1)} = -1,$ $\quad a_{(0,2,1,0)} = 2,$ $\quad a_{(0,2,0,4)} = 1,$ $\quad a_{(0,5,0,0)} = 1$

(3.3.5) Find the cardinality of I_n^m.

As I explained in section, we can use the combinatorial technique of "Stars and bars" to show that the cardinality is

$$\binom{m+n-1}{m} = \binom{m+n-1}{n-1} = \frac{(m+n-1)!}{m!\,(n-1)!}$$

You can also prove this recursively using induction, or even get a good guess for the formula by computing lots of examples.

(3.3.9) Consider the function

$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{cases} \dfrac{x^2y(x-y)}{x^2+y^2} = \dfrac{x^3y-x^2y^2}{x^2+y^2} & \text{if } \begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} \\ 0 & \text{if } \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \end{cases}$$

(a) Compute D_1f and D_2f. Is f of class C^1?

For $\begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$, we compute

$$D_1f\begin{pmatrix}x\\y\end{pmatrix} = \frac{(x^2+y^2)(3x^2y-2xy^2)-(x^3y-x^2y^2)2x}{(x^2+y^2)^2}$$

$$D_2f\begin{pmatrix}x\\y\end{pmatrix} = \frac{(x^2+y^2)(x^3-2x^2y)-(x^3y-x^2y^2)2y}{(x^2+y^2)^2}$$

At $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, we use the definition of partial derivative, Defn. 1.7.3 on page 122:

$$D_1f\begin{pmatrix}0\\0\end{pmatrix} = \lim_{h\to 0}\frac{1}{h}\left(f\begin{pmatrix}h\\0\end{pmatrix} - f\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{0}{h^2} - 0\right) = 0$$

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Now we need to show that f is C^1 by showing that D_1f and D_2f are continuous. Clearly they are continuous at points away from the origin, so we only need to check continuity at $\begin{pmatrix}0\\0\end{pmatrix}$. These limits are much easier in polar coordinates:

$$\lim_{\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}0\\0\end{pmatrix}} D_1f\begin{pmatrix}x\\y\end{pmatrix} = \lim_{r\to 0}\frac{r^5(\text{function of }\theta)}{r^4} = \lim_{r\to 0} r(\text{function of }\theta) = 0$$

and likewise for D_2f. Thus f is C^1.

(b) Show that all second partial derivatives of f exist everywhere.

For $\begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$, we can take all second partial derivaties (we're not asked to compute them explicitly, but we definitely can find them). At $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$,

$$D_1D_1f\begin{pmatrix}0\\0\end{pmatrix} = \lim_{h\to 0}\frac{1}{h}\left(D_1f\begin{pmatrix}h\\0\end{pmatrix} - D_1f\begin{pmatrix}0\\0\end{pmatrix}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{0}{h^4}-0\right) = 0$$

$$D_2D_2f\begin{pmatrix}0\\0\end{pmatrix} = \lim_{h\to 0}\frac{1}{h}\left(D_2f\begin{pmatrix}0\\h\end{pmatrix} - D_2f\begin{pmatrix}0\\0\end{pmatrix}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{0}{h^4}-0\right) = 0$$

(continued) →

(3.3.9 (b) continued)

and

$$D_1 D_2 f\begin{pmatrix}0\\0\end{pmatrix} = \lim_{h\to 0} \frac{1}{h}\left(D_2 f\begin{pmatrix}h\\0\end{pmatrix} - D_2 f\begin{pmatrix}0\\0\end{pmatrix}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{h^5}{h^4} - 0\right) = 1$$

$$D_2 D_1 f\begin{pmatrix}0\\0\end{pmatrix} = \lim_{h\to 0} \frac{1}{h}\left(D_1 f\begin{pmatrix}0\\h\end{pmatrix} - D_1 f\begin{pmatrix}0\\0\end{pmatrix}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{0}{h^4} - 0\right) = 0$$

Thus all second partial derivatives exist everywhere.

(c) Show that $D_1 D_2 f\begin{pmatrix}0\\0\end{pmatrix} \neq D_2 D_1 f\begin{pmatrix}0\\0\end{pmatrix}$.

We computed these at the top of the page:

$$D_1 D_2 f\begin{pmatrix}0\\0\end{pmatrix} = 1 \neq 0 = D_2 D_1 f\begin{pmatrix}0\\0\end{pmatrix}$$

(d) Why doesn't this contradict Theorem 3.3.9?

We showed that $D_1 f$ and $D_2 f$ are continuous, but we never showed that they were differentiable. Theorem 3.3.9 requires that the partial derivatives be differentiable in order to guarantee the equality of mixed partial derivatives.

(3.3.13) Find the Taylor polynomial of degree 2 of the function

$$f(x,y) = (x+y+xy)^{1/2} \quad \text{at the point } (-2,-3)$$

First we compute all of the partial derivatives up to order 2:

$f(x,y) = (x+y+xy)^{1/2}$

$D_{(1,0)}f(x,y) = \frac{1}{2}(x+y+xy)^{-1/2}(1+y)$

$D_{(0,1)}f(x,y) = \frac{1}{2}(x+y+xy)^{-1/2}(1+x)$

$D_{(2,0)}f(x,y) = -\frac{1}{4}(1+y)(x+y+xy)^{-3/2}(1+y)$

$D_{(0,2)}f(x,y) = -\frac{1}{4}(1+x)(x+y+xy)^{-3/2}(1+x)$

$D_{(1,1)}f(x,y) = \frac{1}{2}(x+y+xy)^{-1/2} - \frac{1}{4}(1+y)(x+y+xy)^{-3/2}(1+x)$

Evaluating at $(-2,-3)$, we get

$f(-2,-3) = 1$

$D_{(1,0)}f(-2,-3) = -1$

$D_{(0,1)}f(-2,-3) = -\frac{1}{2}$

$D_{(2,0)}f(-2,-3) = -1$

$D_{(0,2)}f(-2,-3) = -\frac{1}{4}$

$D_{(1,1)}f(-2,-3) = 0$

Combining these and using Definition 3.3.14, we get

$$P^2_{f,\left(\begin{smallmatrix}-2\\-3\end{smallmatrix}\right)}\left(\begin{pmatrix}-2\\-3\end{pmatrix}+\vec{h}\right) = 1 - h_1 - \frac{1}{2}h_2 - \frac{1}{2}h_1^2 - \frac{1}{8}h_2^2$$