Problem Set #8

(3.5.2) Consider the quadratic form from example 3.5.7: \( Q(x) = xy - xz + yz \). Decompose \( Q(x) \) with a different choice of \( u \), to support the statement \( u = x - y \) was not a magical choice.

Let's use \( u = y - z \) instead (lots of choices are possible). Then \( y = u + z \), so

\[
Q(x) = x(u+z) - xz + (u+z)z
= xu + xz - xz + uz + z^2
= xu + uz + z^2
= xu - \left(\frac{u}{2}\right)^2 + \left(\frac{u}{2}\right)^2 + uz + z^2
= xu - \frac{u^2}{4} + \left( z + \frac{u}{2} \right)^2
= \left(\frac{u}{2} - x\right)^2 + z^2 + \left( z + \frac{u}{2} \right)^2
= \left(\frac{u}{2} - x\right)^2 + z^2 + \left( \frac{y}{2} - \frac{z}{2} \right)^2
\]

The exact linear functions in our decomposition are different, but note that the signature doesn't change.
(3.55) What is the signature of the following quadratic forms?
(a) $x^2 + xy$ on $\mathbb{R}^3$

\[
x^2 + xy = x^2 + x y + \left( \frac{y}{2} \right)^2 - \left( \frac{y}{2} \right)^2
= \left( x + \frac{y}{2} \right)^2 - \left( \frac{y}{2} \right)^2
\]
Signature is $(1, 1)$.

(b) $xy + yz$ on $\mathbb{R}^3$

Use the substitution $u = y - x$ so that $y = u + x$. Then

\[
xy + yz = x(u + x) + (u + x)z
= x^2 + xu + zu + xz
= x^2 + x(u + z) + zu
= x^2 + x(u + z) + \left( \frac{u}{2} + \frac{z}{2} \right)^2 - \left( \frac{u}{2} + \frac{z}{2} \right)^2 + zu
= \left( x + \frac{u}{2} + \frac{z}{2} \right)^2 - \left( \frac{u}{4} + \frac{z}{4} \right)^2 + \frac{zu}{2}
= \left( x + \frac{u}{2} + \frac{z}{2} \right)^2 - \left( \frac{u - z}{2} \right)^2
= \left( x + \frac{u}{2} - \frac{z}{2} \right)^2 - \left( \frac{u - z}{2} \right)^2
= \left( x + \frac{u}{2} + \frac{z}{2} \right)^2 - \left( \frac{u - z}{2} - z \right)^2
\]
Signature is $(1, 1)$. 
(3.5.8) Identify \((a, b, c) \in \mathbb{R}^3\) with the upper triangular matrix \(M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\).

(a) What is the signature of the quadratic form \(Q(M) = \text{tr}(M^2)\)? What kind of surface do you get in \(\mathbb{R}^3\) by setting \(\text{tr}(M^2) = 1\)?

\[
M^2 = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a^2 & b(a+d) \\ 0 & d^2 \end{bmatrix}, \text{ and the trace } \text{tr}(M^2) = a^2 + d^2.
\]

See page 75 for the definition of trace, if needed.

So \(\text{tr}(M^2) = a^2 + d^2\) has signature \((2, 0)\). If we set \(\text{tr}(M^2) = a^2 + d^2 = 1\), this is a cylinder opening along the \(b\)-axis, with radius \(1\).

(b) What is the signature of the quadratic form \(Q(M) = \text{tr}(M^TM)\)? What kind of surface do you get by setting \(\text{tr}(M^TM) = 1\)?

Note \(M^TM = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a^2 & ad \\ bd & b^2 + d^2 \end{bmatrix}\), so \(\text{tr}(M^TM) = a^2 + b^2 + d^2\).

This has signature \((3, 0)\).

If we set \(\text{tr}(M^TM) = a^2 + b^2 + d^2 = 1\), this is the unit sphere centered at the origin.
Consider the vector space of $2 \times 2$ matrices such that
\[ H = \begin{bmatrix} a & b + ic \\ b - ic & d \end{bmatrix} \]
where $a, b, c, d$ are real numbers. What is the signature of the quadratic form $Q(H) = \det H$?

\[
\det H = ad - (b + ic)(b - ic) = ad - b^2 - c^2
\]

Let $u = a - d$ so that $a = u + d$. Then
\[
ad - b^2 - c^2 = (u + d)d - b^2 - c^2
= d^2 + ud - b^2 - c^2
= d^2 + ud + \left(\frac{u}{2}\right)^2 - \left(\frac{u}{2}\right)^2 - b^2 - c^2
= \left(d + \frac{u}{2}\right)^2 - \left(\frac{u}{2}\right)^2 - b^2 - c^2
\]

This has signature $(1, 3)$. 

Let $V$ be a vector space. A symmetric bilinear function on $V$ is a mapping $B : V \times V \to \mathbb{R}$ such that

1. $B(\alpha \vec{v} + \beta \vec{w}, \vec{w}) = \alpha B(\vec{v}, \vec{w}) + \beta B(\vec{w}, \vec{w})$ for all $\vec{v}, \vec{w} \in V$ and $\alpha, \beta \in \mathbb{R}$.
2. $B(\vec{v}, \vec{w}) = B(\vec{w}, \vec{v})$ for all $\vec{v}, \vec{w} \in V$.

(a) Show that if $A$ is a symmetric matrix, the mapping $B_A(\vec{v}, \vec{w}) = \vec{v}^T A \vec{w}$ is a symmetric bilinear function.

$$B_A(\alpha \vec{v} + \beta \vec{w}, \vec{w}) = (\alpha \vec{v} + \beta \vec{w})^T A \vec{w}$$
$$= \alpha \vec{v}^T A \vec{w} + \beta \vec{w}^T A \vec{w} = \alpha B_A(\vec{v}, \vec{w}) + \beta B_A(\vec{w}, \vec{w})$$

$$B_A(\vec{v}, \vec{w}) = \vec{v}^T A \vec{w}$$ is a number (check dimensions if you like), so $(\vec{v}^T A \vec{w})^T = \vec{w}^T A^\top \vec{v}$.
But $(\vec{v}^T A \vec{w})^T = \vec{w}^T A^\top \vec{v} = \vec{w}^T A^\top \vec{v}$ (since $A$ is symmetric) = $B_A(\vec{w}, \vec{v})$.

Thus $B_A$ satisfies both properties of a symmetric bilinear function.

(b) Show that every symmetric bilinear function on $\mathbb{R}^n$ is of the form $B_A$ for a unique symmetric matrix $A$.

Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric bilinear function. If there is a matrix $A$ such that $B(\vec{v}, \vec{w}) = \vec{v}^T A \vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$, let's figure out what it would have to be.

Letting $\vec{v} = \vec{e}_i$ and $\vec{w} = \vec{e}_j$ (standard basis vectors), note that

$$\vec{e}_i^T A = \text{[i}^{\text{th}} \text{row of } A]$$

so $\vec{e}_i^T A \vec{e}_j$ is the $i^{\text{th}}$ entry of the $i^{\text{th}}$ row of $A$, call it $a_{ij}$.
This holds for all $i, j$, so the $i^{\text{th}} j^{\text{th}}$ entry of $A$ is

$$a_{ij} = B(\vec{e}_i, \vec{e}_j)$$

This gives us a candidate matrix $A = (a_{ij})$. By construction, $A$ is the unique matrix that at least works on all the basic vectors.

Now check that $A$ works $\forall \vec{v}, \vec{w} \in \mathbb{R}^n$. Write $\vec{v} = \sum_i \vec{v}_i \vec{e}_i$ and $\vec{w} = \sum_j \vec{w}_j \vec{e}_j$ for constants $\vec{v}_i, \vec{w}_j \in \mathbb{R}$. Then

$$\vec{v}^T A \vec{w} = (\sum_i \vec{v}_i \vec{e}_i) (\sum_j \vec{w}_j \vec{e}_j) = \sum_{i,j} \vec{v}_i \vec{w}_j \vec{e}_i^T A \vec{e}_j$$
$$= \sum_i \sum_j \vec{v}_i \vec{w}_j B(\vec{e}_i, \vec{e}_j)$$
$$= B(\sum_i \vec{v}_i \vec{e}_i, \sum_j \vec{w}_j \vec{e}_j) = B(\vec{v}, \vec{w}).$$
This proves the claim $\forall \vec{v}, \vec{w} \in \mathbb{R}^n$, not just the basis vectors.
(3.5.13 continued)

(c) Let $P_k$ be the space of polynomials of degree at most $k$. Show that the function $B: P_k \times P_k \rightarrow \mathbb{R}$ given by $B(p, q) = \int_0^1 p(t)q(t)dt$ is a symmetric bilinear function.

Let $p(t), p_0(t)$, and $q(t)$ be polynomials of degree at most $k$, and let $a, b \in \mathbb{R}$. Then

$$B(ap + bp_0, q) = \int_0^1 [ap(t) + bp_0(t)]q(t)dt$$

$$= \int_0^1 ap(t)q(t)dt + \int_0^1 bp_0(t)q(t)dt$$

$$= a \int_0^1 p(t)q(t)dt + b \int_0^1 p_0(t)q(t)dt$$

$$= aB(p, q) + bB(p_0, q).$$

Also, $B(p, q) = \int_0^1 p(t)q(t)dt = \int_0^1 q(t)p(t)dt = B(q, p)$.

Thus, $B$ is a symmetric bilinear function.

(d) Denote by $p_i(t) = 1, p_s(t) = t, \ldots, p_{k+1}(t) = t^k$ the usual basis of $P_k$, and by $\mathbb{E}_i$, the corresponding "concrete to abstract" linear transformation. Show that $B(\mathbb{E}_i, \mathbb{E}_j)$ is a symmetric bilinear function on $\mathbb{R}^{k+1}$, and find its matrix.

Here $\mathbb{R}^{k+1}$ is $\mathbb{R}^{k+1}$, and recall that $\mathbb{E}_i = e_i^1 = p_i(t)$. Then let $a, b \in \mathbb{R}$ and $q(t), r(t)$, and $s(t)$ be in $P_k$. Let

$$\mathbf{a} = \sum_{i=1}^{k+1} a_i e_i, \quad \mathbf{b} = \sum_{i=1}^{k+1} b_i e_i, \quad \mathbf{c} = \sum_{i=1}^{k+1} c_i e_i$$

and let $d, e \in \mathbb{R}$. Then

$$B(\mathbb{E}_i, \mathbb{E}_j) = B(\mathbb{E}_i, \mathbb{E}_j) = B(\sum_{i=1}^{k+1} a_i p_i(t), \sum_{i=1}^{k+1} b_i p_i(t))$$

$$= d B(\sum_{i=1}^{k+1} a_i p_i(t), \sum_{i=1}^{k+1} c_i p_i(t)) + e B(\sum_{i=1}^{k+1} b_i p_i(t), \sum_{i=1}^{k+1} c_i p_i(t))$$

This shows property 1 of being a symmetric bilinear function.

For property 2,

$$B(\mathbb{E}_i, \mathbb{E}_j) = B(\sum_{i=1}^{k+1} a_i p_i(t), \sum_{i=1}^{k+1} b_i p_i(t))$$

$$= \sum_{i=1}^{k+1} a_i p_i(t), \sum_{i=1}^{k+1} b_i p_i(t) = B(\mathbb{E}_j, \mathbb{E}_i)$$

This completes the proof.
(3.6) (a) Show that \( f(\frac{x}{y}) = x^2 + xy + z^2 - \cos y \) has a critical point at the origin.

Calculate \( Df(\frac{x}{y}) = 2x + y \)
\( Df_x(\frac{x}{y}) = x + \sin y \)
\( Df_y(\frac{x}{y}) = 2z \)

All of these vanish at the origin, so the origin is a critical point.

(b) What kind of critical point does it have?

We calculate the quadratic form associated to \( f \) by finding only the second order Taylor approximation:
\[
\begin{align*}
D(0,0,0) f(\frac{x}{y}) &= 2 \\
D(0,0,0) f\left(\frac{x}{y}\right) &= \cos y \\
D(0,0,0) f\left(\frac{x}{y}\right) &= 2 \\
D(0,0,0) f\left(\frac{x}{y}\right) &= 1 \\
D(0,0,0) f\left(\frac{x}{y}\right) &= 0 \\
D(0,0,0) f\left(\frac{x}{y}\right) &= 0 \\
\end{align*}
\]

Thus \( Q_x(\frac{x}{y}) = h_x^2 + 2h_xh_y + h_y^2 + 1h_xh_2 \)
\[
= h_x^2 + h_xh_1 + (h_y)^2 - (\frac{h_x}{h_y})^2 + \frac{1}{h_y}h_x + h_y^2 \\
= (h_x + \frac{h_x}{h_y})^2 + (h_y)^2 + h_y^2 \\
\]

Since the signature is \((3,0)\), this quadratic form is positive definite, so the origin is a local minimum of \( f \).
(3.6.2)(a) Find the critical points of the function 
\[ f(y) = x^3 - 12xy + 8y^3 \]

Compute 
\[ D_x f(y) = 3x^2 - 12y \]
\[ D_y f(y) = -12x + 24y^2 \]

For these to both be zero, we have \[ 12x = 24y^2, \] so \[ x = 2y^2. \] Substituting in, we get 
\[ 0 = 3(2y^2)^2 - 12y \]
\[ = 12y^4 - 12y \]
\[ = 12y(y^3 - 1) \]
\[ = 12y(y - 1)(y^2 + y + 1) \]

So \[ y = 0 \] or \[ y = 1 \]

If \[ y = 0, \] then \[ x = 0, \] so \[ (0,0) \] is a critical point. If \[ y = 1, \] then \[ x = 2, \] so \[ (2,1) \] is a critical point.

(b) Determine the nature of each critical point.

\[ D_{x,0} f(y) = 6x \]
\[ D_{x,1} f(y) = -12 \]
\[ D_{x,x} f(y) = 118y \]

Then at \( (0,0), \) the quadratic form is \(-6h_2^2, \) which has signature \( (0,1), \) so this is a local maximum.

At \( (2,1), \) the quadratic form is 
\[ 6h_1^3 - 6h_1^2 - 48h_1h_2 \]
\[ = 6(h_1^3 - 4h_1h_2 + 16h_2^2 - 16h_2^2) - 6h_2^2 \]
\[ = 6(h_1 - 4h_2)^2 - 103h_2^2 \]
\[ = \left( \frac{h_1 - 4h_2}{\sqrt{6}} \right)^2 - \left( \frac{h_2}{\sqrt{66}} \right)^2 \]

The signature is \( (1,1), \) so this is a saddle point.

It is entirely possible I made a mistake here. There are a lot of crossed marks,
(b) Find the critical points of the function \( f(x) = xy + yz - xz + xyz \)

First we calculate
\[
D_x f(x) = y - z + yz \\
D_y f(x) = x + z + xz \\
D_z f(x) = y - x + xy
\]

Setting these equal to zero, we get
\[
y - z + yz = 0 \quad \Rightarrow \quad y(1 + z) = z \\
\Rightarrow \quad y = \frac{z}{1 + z} \\
x + z + xz = 0 \quad \Rightarrow \quad x(1 + z) = -z \\
\Rightarrow \quad x = \frac{-z}{1 + z}
\]

(Unless \(z = -1\), which leads to a contradiction)

Plugging these into the third equation, we get
\[
\frac{z}{1 + z} + \frac{z}{1 + z} - \frac{z^2}{(1 + z)^2} = 0
\]
\[
\Rightarrow \quad 2z(2z + 1) - z^2 = 0
\]
\[
\Rightarrow \quad z^2 + 2z = 0
\]

Thus \(z = 0\) or \(z = -2\).

If \(z = 0\) we get \( (0, 0, 0) \) is a critical point. If \(z = -2\), we get \( (0, 0, -2) \) is a critical point.

(b) Determine the nature of each critical point.

\[
D(0,0,0) f(x) = 0 \\
D(0,0,0) f(x) = 0 \\
D(0,0,0) f(x) = 0 \\
D(0,0,0) f(x) = 1 + z \\
D(0,0,0) f(x) = -1 + y \\
D(0,0,0) f(x) = 1 + x
\]

Thus at \( (0,0,0) \) the quadratic form is \( h_{11} - h_{22} + h_{33} \). From example 3.5.7, we know that this has signature \((2,1)\), so \((0,0,0)\) is a saddle point.

At \((0,0,0)\), the quadratic form is \(-h_{11} + h_{22} + h_{33}\). This is just the negative of the other quadratic form, so has signature \((1,3)\) so \((0,0,0)\) is also a saddle point.