Problem Set #8

(3.5.2) Consider the quadratic form from example 3.5.7: $Q(\vec{x}) = xy - xz + yz$. Decompose $Q(\vec{x})$ with a different choice of u, to support the statement $u = x-y$ was not a magical choice.

Let's use $u = y-z$ instead (lots of choices are possible). Then $y = u+z$, so

$$\begin{aligned}
Q(\vec{x}) &= x(u+z) - xz + (u+z)z \\
&= xu + xz - xz + uz + z^2 \\
&= xu + uz + z^2 \\
&= xu - (\tfrac{u}{2})^2 + (\tfrac{u}{2})^2 + uz + z^2 \\
&= xu - \tfrac{u^2}{4} + (z + \tfrac{u}{2})^2 \\
&= -(\tfrac{u^2}{4} - xu) + (z + \tfrac{u}{2})^2 \\
&= -(\tfrac{u^2}{4} - xu + x^2 - x^2) + (z + \tfrac{u}{2})^2 \\
&= -(\tfrac{u}{2} - x)^2 + x^2 + (z + \tfrac{u}{2})^2 \\
&= -(\tfrac{y}{2} - \tfrac{z}{2} - x)^2 + x^2 + (\tfrac{y}{2} - \tfrac{z}{2})^2
\end{aligned}$$

The exact linear functions in our decomposition are different, but note that the signature doesn't change.

(3.5.5) What is the signature of the following quadratic forms?

(a) $x^2 + xy$ on $\mathbb{R}^2$

$$x^2 + xy = x^2 + xy + \left(\tfrac{y}{2}\right)^2 - \left(\tfrac{y}{2}\right)^2$$
$$= \left(x + \tfrac{y}{2}\right)^2 - \left(\tfrac{y}{2}\right)^2$$

Signature is $(1,1)$

(b) $xy + yz$ on $\mathbb{R}^3$

Use the substitution $u = y - x$ so that $y = u + x$. Then

$$\begin{aligned}
xy + yz &= x(u+x) + (u+x)z \\
&= x^2 + xu + zu + xz \\
&= x^2 + x(u+z) + zu \\
&= x^2 + x(u+z) + \left(\tfrac{u+z}{2}\right)^2 - \left(\tfrac{u+z}{2}\right)^2 + zu \\
&= \left(x + \tfrac{u}{2} + \tfrac{z}{2}\right)^2 - \tfrac{u^2}{4} - \tfrac{z^2}{4} - \tfrac{uz}{2} + zu \\
&= \left(x + u/2 + z/2\right)^2 - \tfrac{u^2}{4} - \tfrac{z^2}{4} + \tfrac{uz}{2} \\
&= \left(x + u/2 + z/2\right)^2 - \left(\tfrac{u-z}{2}\right)^2 \\
&= \left(x + \tfrac{y}{2} - \tfrac{x}{2} + \tfrac{z}{2}\right)^2 - \left(\tfrac{y}{2} - \tfrac{x}{2} - \tfrac{z}{2}\right)^2 \\
&= \left(\tfrac{x}{2} + \tfrac{y}{2} + z/2\right)^2 - \left(y/2 - x/2 - z/2\right)^2
\end{aligned}$$

Signature is $(1,1)$.

(3.5.8) Identify $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ with the upper triangular matrix $M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

(a) What is the signature of the quadratic form $Q(M) = tr(M^2)$? What kind of surface do you get in $\mathbb{R}^3$ by setting $tr(M^2) = 1$?

$$M^2 = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a^2 & b(a+d) \\ 0 & d^2 \end{bmatrix}$$, and the trace $tr(M^2) = a^2 + d^2$.

See page 75 for the definition of trace, if needed.

So $tr(M^2) = a^2 + d^2$ has signature $(2,0)$. If we set $tr(M^2) = a^2 + d^2 = 1$, this is a cylinder opening along the b-axis, with radius 1.

(b) What is the signature of the quadratic form $Q(M) = tr(M^T M)$? What kind of surface do you get by setting $tr(M^T M) = 1$?

Note $M^T M = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a^2 & ad \\ ba & b^2+d^2 \end{bmatrix}$, so $tr(M^T M) = a^2 + b^2 + d^2$

This has signature $(3,0)$

If we set $tr(M^T M) = a^2 + b^2 + d^2 = 1$, this is the unit sphere centered at the origin.

(3.5.9) Consider the vector space of 2×2 matrices such that

$$H = \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$$

where a,b,c,d are real numbers. What is the signature of the quadratic form $Q(H) = \det H$?

$$\det H = ad - (b+ic)(b-ic) = ad - b^2 - c^2$$

Let $u = a-d$ so that $a = u+d$. Then

$$\begin{aligned} ad - b^2 - c^2 &= (u+d)d - b^2 - c^2 \\ &= d^2 + ud - b^2 - c^2 \\ &= d^2 + ud + (\tfrac{u}{2})^2 - (\tfrac{u}{2})^2 - b^2 - c^2 \\ &= (d + \tfrac{u}{2})^2 - (\tfrac{u}{2}) - b^2 - c^2 \\ &= (\tfrac{a+d}{2})^2 - (\tfrac{a-d}{2})^2 - b^2 - c^2 \end{aligned}$$

This has signature $(1,3)$.

(3.5.13) Let V be a vector space. A symmetric bilinear function on V is a mapping $B: V \times V \to \mathbb{R}$ such that

(1) $B(a\bar{v}_1 + b\bar{v}_2, \bar{w}) = aB(\bar{v}_1, \bar{w}) + bB(\bar{v}_2, \bar{w})$ for all $\bar{v}_1, \bar{v}_2, \bar{w} \in V$ and $a, b \in \mathbb{R}$.

(2) $B(\bar{v}, \bar{w}) = B(\bar{w}, \bar{v})$ for all $\bar{v}, \bar{w} \in V$

(a) Show that if A is a symmetric matrix, the mapping $B_A(\bar{v}, \bar{w}) = \bar{v}^T A \bar{w}$ is a symmetric bilinear function.

$$\begin{aligned} B_A(a\bar{v}_1 + b\bar{v}_2, \bar{w}) &= (a\bar{v}_1 + b\bar{v}_2)^T A \bar{w} \\ &= (a\bar{v}_1^T + b\bar{v}_2^T) A \bar{w} \\ &= a\bar{v}_1^T A \bar{w} + b\bar{v}_2^T A \bar{w} = aB_A(\bar{v}_1, \bar{w}) + bB_A(\bar{v}_2, \bar{w}) \end{aligned}$$

$B_A(\bar{v}, \bar{w}) = \bar{v}^T A \bar{w}$ is a number (check dimensions if you like), so $(\bar{v}^T A \bar{w})^T = \bar{v}^T A \bar{w}$.
But $(\bar{v}^T A \bar{w})^T = \bar{w}^T A^T \bar{v} = \bar{w}^T A^T \bar{v}$ (since A is symmetric) $= B_A(\bar{w}, \bar{v})$.

Thus B_A satisfies both properties of a symmetric bilinear function.

(b) Show that every symmetric bilinear function on $\mathbb{R}^n$ is of the form B_A for a unique symmetric matrix A.

Let $B: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric bilinear function. If there is a matrix A such that $B(\bar{v}, \bar{w}) = \bar{v}^T A \bar{w}$ for all $\bar{v}, \bar{w} \in \mathbb{R}^n$, let's figure out what it would have to be.

Letting $\bar{v} = \bar{e}_i$ and $\bar{w} = \bar{e}_j$ (standard basis vectors), note that

$$\bar{e}_i^T A = [i^{th} \text{ row of } A]$$

so $\bar{e}_i^T A \bar{e}_j$ = the j^{th} entry of the i^{th} row of A, call it a_{ij}.

This holds for all i, j, so the i,j^{th} entry of A is

$$a_{ij} = B(\bar{e}_i, \bar{e}_j)$$

This gives us a candidate matrix $A = (a_{ij})$. By construction, it is the unique matrix that at least works on all the basis vectors.

Now check that A works $\forall\, \bar{v}, \bar{w} \in \mathbb{R}^n$. Write $\bar{v} = \sum_{i=1}^n v_i \bar{e}_i$ and $\bar{w} = \sum_{j=1}^n w_j \bar{e}_j$ for constants $v_i, w_j \in \mathbb{R}$. Then

$$\begin{aligned} \bar{v}^T A \bar{w} = \left(\sum_{i=1}^n v_i \bar{e}_i^T\right) A \left(\sum_{j=1}^n w_j \bar{e}_j\right) &= \sum_{i=1}^n \sum_{j=1}^n v_i w_j \bar{e}_i^T A \bar{e}_j \\ &= \sum_{i,j}\sum v_i w_j B(\bar{e}_i, \bar{e}_j) \\ &= \sum\sum B(v_i \bar{e}_i, w_j \bar{e}_j) = B\left(\sum_i v_i \bar{e}_i, \sum_j w_j \bar{e}_j\right) = B(\bar{v}, \bar{w}). \end{aligned}$$

This proves the claim $\forall\, \bar{v}, \bar{w} \in \mathbb{R}^n$, not just the basis vectors,

(continued) →

(3.5.13 continued)

(c) Let P_k be the space of polynomials of degree at most k. Show that the function $B: P_k \times P_k \to \mathbb{R}$ given by $B(p,q) = \int_0^1 p(t)q(t)\,dt$ is a symmetric bilinear function.

Let $p_1(t)$, $p_2(t)$, and $q(t)$ be polynomials of degree at most k, and let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} B(ap_1 + bp_2, q) &= \int_0^1 (ap_1(t) + bp_2(t))q(t)\,dt \\ &= \int_0^1 ap_1(t)q(t)\,dt + \int_0^1 bp_2(t)q(t)\,dt \\ &= a\int_0^1 p_1(t)q(t)\,dt + b\int_0^1 p_2(t)q(t)\,dt \\ &= aB(p_1, q) + bB(p_2, q). \end{aligned}$$

Also, $B(p,q) = \int_0^1 p(t)q(t)\,dt = \int_0^1 q(t)p(t)\,dt = B(q,p)$.

Thus, B is a symmetric bilinear function.

(d) Denote by $p_1(t) = 1$, $p_2(t) = t$, $\ldots$, $p_{k+1}(t) = t^k$ the usual basis of P_k, and by Φ_p the corresponding "concrete to abstract" linear transformation. Show that $B(\Phi_p(\bar{a}), \Phi_p(\bar{b}))$ is a symmetric bilinear function on $\mathbb{R}^n$, and find its matrix.

Here $\mathbb{R}^n$ is $\mathbb{R}^{k+1}$, and recall that $\Phi_p(\bar{e}_i) = t^{i-1} = p_i(t)$. Then let $a, b \in \mathbb{R}$ and $q(t)$, $r(t)$, and $s(t)$ be in P_k. Let

$$\bar{a} = \sum_{i=1}^{k+1} a_i\bar{e}_i, \quad \bar{b} = \sum_{j=1}^{k+1} b_j\bar{e}_j, \quad \bar{c} = \sum_{\ell=1}^{k+1} c_\ell\bar{e}_\ell$$

and let $d, e \in \mathbb{R}$. Then

$$\begin{aligned} B(\Phi_p(d\bar{a} + e\bar{b}), \Phi(\bar{c})) &= B\Big(\sum_{i=1}^{k+1}(da_i + eb_i)p_i(t), \sum_{\ell=1}^{k+1} c_\ell p_\ell(t)\Big) \\ &= B\Big(\sum_{i=1}^{k+1} da_i p_i(t), \sum_{\ell=1}^{k+1} c_\ell p_\ell(t)\Big) + B\Big(\sum_{i=1}^{k+1} eb_i p_i(t), \sum_{\ell=1}^{k+1} c_\ell p_\ell(t)\Big) \\ &= dB\Big(\sum_{i=1}^{k+1} a_i p_i(t), \sum_{\ell=1}^{k+1} c_\ell p_\ell(t)\Big) + eB\Big(\sum_{i=1}^{k+1} b_i p_i(t), \sum_{\ell=1}^{k+1} c_\ell p_\ell(t)\Big) \\ &= dB(\Phi_p(\bar{a}), \Phi_p(\bar{c})) + eB(\Phi_p(\bar{b}), \Phi_p(\bar{c})) \end{aligned}$$

This shows property 1 of being a symmetric bilinear function.

For property 2,

$$\begin{aligned} B(\Phi_p(\bar{a}), \Phi_p(\bar{b})) &= B\Big(\sum_{i=1}^{k+1} a_i p_i(t), \sum_{j=1}^{k+1} b_j p_j(t)\Big) \\ &= B\Big(\sum_{j=1}^{k+1} b_j p_j(t), \sum_{i=1}^{k+1} a_i p_i(t)\Big) = B(\Phi_p(\bar{b}), \Phi_p(\bar{a})) \end{aligned}$$

This completes the proof.

(3.6.1)(a) Show that $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + xy + z^2 - \cos y$ has a critical point at the origin.

Calculate
$$D_1 f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x + y$$
$$D_2 f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + \sin y$$
$$D_3 f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2z$$

All of these vanish at the origin, so the origin is a critical point.

(b) What kind of critical point does it have?

We calculate the quadratic form associated to f by finding only the second order Taylor approximation:

$$D_{(2,0,0)} f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2$$
$$D_{(0,2,0)} f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \cos y$$
$$D_{(0,0,2)} f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2$$
$$D_{(1,1,0)} f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$$
$$D_{(1,0,1)} f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$
$$D_{(0,1,1)} f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Thus
$$\begin{aligned} Q_f(\bar{h}) &= h_1^2 + \tfrac{1}{2}h_2^2 + h_3^2 + 1 h_1 h_2 \\ &= h_1^2 + h_1 h_2 + \left(\tfrac{h_2}{2}\right)^2 - \left(\tfrac{h_2}{2}\right)^2 + \tfrac{1}{2}h_2^2 + h_3^2 \\ &= \left(h_1 + \tfrac{h_2}{2}\right)^2 + \left(\tfrac{h_2}{2}\right)^2 + h_3^2 \end{aligned}$$

Since the signature is (3, 0), this quadratic form is positive definite, so the origin is a local minimum of f.

(3.6.2) (a) Find the critical points of the function
$$f\begin{pmatrix}x\\y\end{pmatrix} = x^3 - 12xy + 8y^3$$

Compute $D_1 f\begin{pmatrix}x\\y\end{pmatrix} = 3x^2 - 12y$

$D_2 f\begin{pmatrix}x\\y\end{pmatrix} = -12x + 24y^2$

For these to both be zero, we have $12x = 24y^2$, so $x = 2y^2$. Substituting in, we get
$$\begin{aligned} 0 &= 3(2y^2)^2 - 12y \\ &= 12y^4 - 12y \\ &= 12y(y^3 - 1) \\ &= 12y(y-1)(y^2 + y + 1) \end{aligned}$$

So $y = 0$ or $y = 1$

If $y=0$, then $x=0$, so $(0,0)$ is a critical point. If $y=1$, then $x=2$, so $(2,1)$ is a critical point.

(b) Determine the nature of each critical point.

$D_{(2,0)} f\begin{pmatrix}x\\y\end{pmatrix} = 6x$

$D_{(0,2)} f\begin{pmatrix}x\\y\end{pmatrix} = -12$

$D_{(1,1)} f\begin{pmatrix}x\\y\end{pmatrix} = 48y$

Then at $(0,0)$, the quadratic form is $-6h_2^2$. This has signature $(0,1)$, so this is a local maximum.

At $(2,1)$, the quadratic form is
$$\begin{aligned} & 6h_1^2 - 6h_2^2 - 48h_1h_2 \\ &= 6(h_1^2 - 8h_1h_2 + 16h_2^2 - 16h_2^2) - 6h_2^2 \\ &= 6(h_1 - 4h_2)^2 - 102h_2^2 \\ &= \left(\frac{h_1 - 4h_2}{\sqrt{6}}\right)^2 - \left(\frac{h_2}{\sqrt{102}}\right)^2 \end{aligned}$$

The signature is $(1,1)$, so this is a saddle point.

It is entirely possible I made a mistake here. There are a lot of eraser marks.

(3.6.5) (a) Find the critical points of the function $f\begin{pmatrix}x\\y\\z\end{pmatrix} = xy + yz - xz + xyz$

First we calculate
$$D_1 f\begin{pmatrix}x\\y\\z\end{pmatrix} = y - z + yz$$
$$D_2 f\begin{pmatrix}x\\y\\z\end{pmatrix} = x + z + xz$$
$$D_3 f\begin{pmatrix}x\\y\\z\end{pmatrix} = y - x + xy$$

Setting these equal to zero, we can get
$$y - z + yz = 0 \Rightarrow y(1+z) = z$$
$$\Rightarrow y = \frac{z}{1+z}$$
(unless $z=-1$, which leads to a contradiction
$$x + z + xz = 0 \Rightarrow x(1+z) = -z$$
$$\Rightarrow x = \frac{-z}{1+z}$$

Plugging these into the third equation, we get
$$\frac{z}{1+z} + \frac{z}{1+z} - \frac{z^2}{(1+z)^2} = 0$$
$$\Rightarrow 2z(z+1) - z^2 = 0$$
$$\Rightarrow z^2 + 2z = 0$$
Thus $z = 0$ or $z = -2$.
If $z=0$ we get $\begin{pmatrix}0\\0\\0\end{pmatrix}$ is a critical point. If $z=-2$, we get $\begin{pmatrix}-2\\2\\-2\end{pmatrix}$ is a critical point.

(b) Determine the nature of each critical point.

$$D_{(2,0,0)} f\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$
$$D_{(0,2,0)} f\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$
$$D_{(0,0,2)} f\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$
$$D_{(1,1,0)} f\begin{pmatrix}x\\y\\z\end{pmatrix} = 1 + z$$
$$D_{(1,0,1)} f\begin{pmatrix}x\\y\\z\end{pmatrix} = -1 + y$$
$$D_{(0,1,1)} f\begin{pmatrix}x\\y\\z\end{pmatrix} = 1 + x$$

Thus at $(0,0,0)$ the quadratic form is $h_1h_2 - h_1h_3 + h_2h_3$. From example 3.5.7, we know that this has signature $(2,1)$, so $\begin{pmatrix}0\\0\\0\end{pmatrix}$ is a saddle point.

At $(-2,2,-2)$, the quadratic form is $-h_1h_2 + h_1h_3 - h_2h_3$. This is just the negative of the other quadratic form, so has signature $(1,2)$ so $\begin{pmatrix}-2\\2\\-2\end{pmatrix}$ is also a saddle point.