Problem Set #9

(3.7.2) (a) Show that the function $f\begin{pmatrix}x\\y\\z\end{pmatrix} = x+y+z$ constrained to the surface Y of equation $x = \sin z$ has no critical points.

Let $F(x,y,z) = x - \sin z$ so that Y is given as the zero set of F. Then we calculate

$$\left[Df\begin{pmatrix}x\\y\\z\end{pmatrix}\right] = \begin{bmatrix}1 & 1 & 1\end{bmatrix}$$

$$\left[DF\begin{pmatrix}x\\y\\z\end{pmatrix}\right] = \begin{bmatrix}1 & 0 & -\cos z\end{bmatrix}$$

Then by Theorem 3.7.7, any critical point of f constrained to Y must satisfy the system of equations

$$\begin{aligned} 1 &= \lambda \cdot 1 \\ 1 &= \lambda \cdot 0 \\ 1 &= -\cos z \end{aligned}$$

for some constant λ. But no λ can make the second equation true, so there are no constrained critical points.

(b) Explain geometrically why this is so.

the y-axis

Any critical point $\bar{c}$ must satisfy $T_{\bar{c}}Y \subset \ker[Df(\bar{c})]$. Note that the $x=z=0$ is contained in the tangent space to Y at <u>every</u> point, so

$$\begin{bmatrix}0\\1\\0\end{bmatrix} \in T_{\bar{c}}Y$$

for any $\bar{c}$. However, the kernel of Df is just the vectors lying in the plane $x+y+z=0$. But this does not contain the y-axis.

(3.7.3) (a) Find the critical points of the function $x^3+y^3+z^3$ on the intersection of the planes of equation $x+y+z=2$ and $x+y-z=3$.

Ultimately in part (b) we're going to want to classify the critical points, so our best bet is to find a parametrization for this constraint manifold. The constraint manifold is a line (the intersection of two planes) so this hope is not unrealistic.

Any point on the line is a solution to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 1 & -1 & 3 \end{array}\right]$$

Row reducing, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

Thus the line is given by

$$x = 5/2 - y$$
$$z = -1/2$$

We can rewrite this as the parametrization $\varphi : \mathbb{R} \to \mathbb{R}^3$ sending.

$$y \longmapsto \begin{pmatrix} 5/2 - y \\ y \\ -1/2 \end{pmatrix}$$

Now following Example 3.7.5, the constrained critical points are solutions to $D(f \circ \varphi) = 0$. Here $f\circ\varphi = (5/2 - y)^3 + y^3 + (-\tfrac{1}{2})^3$. Then

$$D(f\circ\varphi) = 3(5/2 - y)^2(-1) + 3y^2$$

Writing this out more explicitly,

$$D(f\circ\varphi) = -3\left(\tfrac{25}{4} - 5y + y^2\right) + 3y^2$$
$$= 15y - 75/4$$

Then $D(f\circ\varphi) = 0$ only when $y = 5/4$, so the only constrained critical point is.

$$\begin{pmatrix} 5/4 \\ 5/4 \\ -1/2 \end{pmatrix}$$

(b) Are the critical points maxima, minima, or neither?

On the constraint manifold, the function whose critical point I'm finding is $(5/2 - y)^3 + (y)^3 + (-\tfrac{1}{2})^3$

$$= \tfrac{15}{2}y^2 - \tfrac{75}{4}y + \left(\tfrac{125}{8} - \tfrac{1}{8}\right)$$

I could use quadratic forms on this, but I already observe that it is a parabola, so the only critical point is a local minima (the parabola opens up).

(3.7.4) (a) Show that the set $X \subset \text{Mat}(2,2)$ of 2×2 matrices with determinant 1 is a smooth submanifold. What is its dimension?

As usual, we identify $\text{Mat}(2,2)$ with $\mathbb{R}^4$ via the map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \leftrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Then X is set of all points in $\mathbb{R}^4$ such that $F\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = ad - bc - 1 = 0$.

Note that F is C^1 and that $[DF] = [d \quad -c \quad -b \quad a]$ is onto since $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \notin X$.

Thus X is a smooth submanifold with dimension 3 (since we have 3 nonpivot columns).

(b) Find a matrix in X which is closest to the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

When we say "closest" we are using the distance function on $\mathbb{R}^4$. Thus we want to minimize

$$\sqrt{a^2 + (b-1)^2 + (c-1)^2 + d^2}$$

However, minimizing the distance function occurs at the same points as minimizing the square of the distance function, so instead I'll minimize

$$f\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a^2 + (b-1)^2 + (c-1)^2 + d^2$$

By Lagrange multipliers, we find that our constrained critical points satisfy

$$[2a \quad 2b-2 \quad 2c-2 \quad 2d] = \lambda[d \quad -c \quad -b \quad a]$$

This gives equations

$$2a = \lambda d \qquad (1)$$
$$2b-2 = -\lambda c \qquad (2)$$
$$2c-2 = -\lambda b \qquad (3)$$
$$2d = \lambda a \qquad (4)$$
$$ad - bc - 1 = 0 \qquad (5)$$

From (1) and (4) we get that $\lambda = \pm 2$. If $\lambda = -2$, then (2) and (3) imply that $2 = -2$, so this is impossible. Thus $\lambda = 2$.

(continued) →

(3.7.4 continued)

With $\lambda = 2$, we got that $a = d$ and $2b + 2c = 2$, so $b + c = 1$. Plugging these into equation (5) gives

$$a^2 - b(1-b) = 1$$

So if I pick a, then b is determined, which determines c, and d is determined by a. So it seems like I have infinitely many critical points, one for each $a \in [-1, 1]$. Let's see that all of these critical points are the same distance from $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

So I am looking at my infinite collection of critical points satisfying

$$a = d$$
$$b + c = 1$$
$$a^2 - b^2 + b = 1$$

Now plug this data into the square of the distance function:

$$\begin{aligned}
a^2 + (b-1)^2 + (c-1)^2 + d^2 &= \underbrace{a^2 + b^2 - b}_{=1} - b + 1 + c^2 - c - c + 1 + d^2 \\
&= 3 \underbrace{- b - c}_{=-1} + c^2 - c + d^2 \\
&= 2 + c^2 - c + d^2 \\
&= 2 + c(c-1) + a^2 \\
&= 2 + (1-b)(-b) + a^2 \\
&= 2 + a^2 + b^2 - b \\
&= 3
\end{aligned}$$

Thus all of the critical points lie distance $\sqrt{3}$ from $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We were only asked to find one of these infinitely many closest points, so take your pick. The most obvious choices are

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

(3.7.5) What is the volume of the largest rectangular parallel aligned with the axes (a box) and contained in the ellipsoid $x^2+4y^2+9z^2 \le 9$.

Our box will have 8 vertices, one in each octant of $\mathbb{R}^3$. The box will extend equal lengths in the $\pm x$ direction, equal lengths in the $\pm y$ direction, and equal lengths in the $\pm z$ direction. Thus the total volume will be 8 times the volume in the octant where each of x, y, z are positive. Thus the total volume is

$$8xyz$$

subject to the constraint that $x^2+4y^2+9z^2=9$.

Hence we maximize $xyz = f$ subject to $F = x^2+4y^2+9z^2-9=0$. Then at our critical point,

$$[yz \quad xz \quad xy] = \lambda[2x \quad 8y \quad 18z]$$

Thus we have the system of equations

$$yz = \lambda \cdot 2x \qquad (1)$$
$$xz = \lambda \cdot 8y \qquad (2)$$
$$xy = \lambda \cdot 18z \qquad (3)$$
$$x^2+4y^2+9z^2=9 \qquad (4)$$

Multiply (1),(2),(3) by x, y, z respectively to get

$$\lambda(2x^2) = \lambda(8y^2) = \lambda(18z^2)$$

Thus either $\lambda = 0$ (in which case $xyz=0$, the minima), or

$$x^2 = 4y^2 = 9z^2.$$

Substituting this into (4) we get

$$3x^2 = 9$$
$$x = \sqrt{3}$$
$$y = \sqrt{3}/2$$
$$z = \sqrt{3}/3$$

Thus the maximum volume is

$$8(\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{3}\right) = \frac{24\sqrt{3}}{6} = \boxed{4\sqrt{3}}$$

The solutions manual says $4\sqrt{2}$. This is incorrect.

(3.7.7) (a) Show that the function xyz has four critical points on the plane of equation $\varphi\begin{pmatrix}x\\y\\z\end{pmatrix} = ax+by+cz-1=0$ when $a,b,c>0$.

Using the hint from the margin, we parametrize the plane by x and y:

$$g:\begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}x\\y\\\frac{1-ax-by}{c}\end{pmatrix}$$

Now we use this paramatrization and follow the method of example 3.7.5. Let $f\begin{pmatrix}x\\y\\z\end{pmatrix} = xyz$. Then

$$(f\circ g) = xy\left(\frac{1-ax-by}{c}\right) = \frac{xy}{c} - \frac{ax^2y}{c} - \frac{bxy^2}{c}$$

The constrained critical points occur precisely when $D_1(f\circ g) = D_2(f\circ g) = 0$.

$$D_1(f\circ g) = \frac{y}{c} - \frac{2axy}{c} - \frac{by^2}{c}$$

$$D_2(f\circ g) = \frac{x}{c} - \frac{ax^2}{c} - \frac{2bxy}{c}$$

Since we want to set these equal to 0, we can multiply through by c, yielding

$$y - 2axy - by^2 = 0 \qquad (1)$$
$$x - ax^2 - 2bxy = 0 \qquad (2)$$

Solving the first for x we get either $y=0$ or

$$x = \frac{y-by^2}{2ay} = \frac{1-by}{2a}$$

Plugging this into (2) we note that either $x=0$ or

$$\frac{1-by}{2a} - a\left(\frac{1-by}{2a}\right)^2 - 2by\left(\frac{1-by}{2a}\right) = 0$$

After a lot of algebra, this yields $y=\frac{1}{3b}$, in which case $x=\frac{1}{3a}$ and $z=\frac{1}{3c}$. If $x=0$ we solve separately, and likewise for $y=0$. In all this yields four constrained critical points:

$$\begin{pmatrix}\frac{1}{3a}\\\frac{1}{3b}\\\frac{1}{3c}\end{pmatrix}, \begin{pmatrix}0\\0\\\frac{1}{c}\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{b}\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{a}\\0\\0\end{pmatrix}$$

(b) Show that three of these critical points are saddles and one is a maximum.

Let's do $\begin{pmatrix}\frac{1}{3a}\\\frac{1}{3b}\\\frac{1}{3c}\end{pmatrix}$ using quadratic forms.

(continued) →

(3.7.7 continued)

We take the second partial derivatives of $(f\circ g)$ and get

$$D_{(2,0)}(f\circ g) = -\frac{2ay}{c}$$
$$D_{(1,1)}(f\circ g) = \frac{1}{c} - \frac{2ax}{c} - \frac{2by}{c}$$
$$D_{(0,2)}(f\circ g) = -\frac{2bx}{c}$$

Plugging in the point $\begin{pmatrix} \frac{1}{3a} \\ \frac{1}{3b} \\ \frac{1}{3c} \end{pmatrix}$ and constructing the quadratic form we get

doesn't appear since z is solved for

$$-\frac{a}{3bc}h_1^2 + \underbrace{\left(\frac{1}{c} - \frac{2}{3c} - \frac{2}{3c}\right)}_{= -\frac{1}{3c}} h_1h_2 - \frac{b}{3ac}h_2^2$$

Now we need to find the signature. Factor out a $-\frac{a}{3bc}$ to get

$$-\frac{a}{3bc}\left(h_1^2 + \frac{b}{a}h_1h_2 + \frac{b^2}{a^2}h_2^2\right)$$

$$= \frac{-a}{3bc}\left(h_1^2 + \frac{b}{a}h_1h_2 + \frac{b^2}{4a^2}h_2^2 - \frac{b^2}{4a^2}h_2^2 + \frac{b^2}{a^2}h_2^2\right)$$

$$= \frac{-a}{3bc}\left(\left(h_1 + \frac{b}{2a}h_2\right)^2 + \frac{b^2}{3a^2}h_2^2\right)$$

$$= \frac{-a}{3bc}\left(\left(h_1 + \frac{b}{2a}h_2\right)^2 + \left(\sqrt{\frac{b^2}{3a^2}}\,h_2\right)^2\right)$$

Then since $a, b, c > 0$, this is negative definite, hence a local maxima.

Without going into the details, suffice it to say that all of

$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{c} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{b} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{a} \\ 0 \\ 0 \end{pmatrix}$$

are saddles, with signature $(1,1)$, which you can check (they should be easier than this one that I did).

(3.7.13) The cone of equation $z^2 = x^2 + y^2$ is cut by the plane $z = 1 + x + y$ in a curve C. Find the points of C closest to and furthest from the origin.

Since we're looking to minimize and maximize distance, we could use the function

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = \sqrt{x^2+y^2+z^2}$$

But note that distance maximizes and minimizes at the same location as at the square of the distance function, so instead I'll use

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = x^2+y^2+z^2$$

Meanwhile, in the notation of Theorem 3.7.7,

$$F_1\begin{pmatrix}x\\y\\z\end{pmatrix} = z^2 - x^2 - y^2$$

$$F_2\begin{pmatrix}x\\y\\z\end{pmatrix} = z - 1 - x - y$$

Now we compute $\left[Df\begin{pmatrix}x\\y\\z\end{pmatrix}\right] = \begin{bmatrix}2x & 2y & 2z\end{bmatrix}$

$$\left[DF_1\begin{pmatrix}x\\y\\z\end{pmatrix}\right] = \begin{bmatrix}-2x & -2y & 2z\end{bmatrix}$$

$$\left[DF_2\begin{pmatrix}x\\y\\z\end{pmatrix}\right] = \begin{bmatrix}-1 & -1 & 1\end{bmatrix}$$

Then any critical points must satisfy

$$2x = -2\lambda_1 x - \lambda_2 \qquad (1)$$

$$2y = -2\lambda_1 y - \lambda_2 \qquad (2)$$

$$2z = 2\lambda_1 z + \lambda_2 \qquad (3)$$

$$z^2 = x^2 + y^2 \qquad (4)$$

$$z = 1 + x + y \qquad (5)$$

From (1), we get $x(2 + 2\lambda_1) = -2\lambda_2$, or $x = \dfrac{-2\lambda_2}{2+2\lambda_1}$

From (2), we get $y = \dfrac{-\lambda_2}{2+2\lambda_1}$

Note that $x = y$, unless $\lambda_1 = -1$ and $\lambda_2 = 0$. This would contradict the other equations (from (3) you would get $z = 0$, which doesn't lie on the intersection of the cone and the plane). Thus $x = y$.

Substituting this into (4) and (5) we get

$$z^2 = 2x^2$$

$$z = 1 + 2x$$

Then $z^2 = 4x^2 + 4x + 1$, so $2x^2 + 4x + 1 = 0$, so $x = \dfrac{-4 \pm \sqrt{16-8}}{4} = \dfrac{-2 \pm \sqrt{2}}{2}$

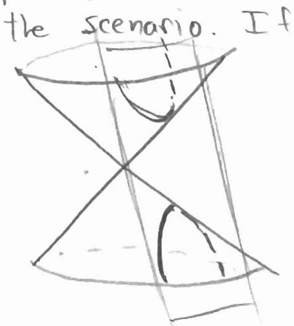
(continued) →

(3.7.13 continued)

Plugging these x back in, we can solve for z to get that the two critical points ar

$$\begin{pmatrix} \frac{-2+\sqrt{2}}{2} \\ \frac{-2+\sqrt{2}}{2} \\ -1+\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{-2-\sqrt{2}}{2} \\ \frac{-2-\sqrt{2}}{2} \\ -1-\sqrt{2} \end{pmatrix}$$

To classify these critical points, we don't have the use of quadratic forms, so we can just use the geometry of the scenario. If we graph the cone and the plane, we get



Note that the cone and plane intersect in a hyperbola, so both of our critical points are local minima of the distance function. Thus the closest point to the origin is

$$\begin{pmatrix} \frac{-2+\sqrt{2}}{2} \\ \frac{-2+\sqrt{2}}{2} \\ -1+\sqrt{2} \end{pmatrix}$$

and there is no furthest point from the origin (it's unbounded).