Math 203A - Solution Set 3

Problem 1. (i) *Four points in* $\mathbb{P}^2$ *are said to be in general position if no three are collinear (i.e. lie on a projective line in the projective plane). Show that if* $p_1, \ldots, p_4$ *are points in general position, there exists a linear change of coordinates*

$$T : \mathbb{P}^2 \to \mathbb{P}^2$$

with

$$T([1:0:0]) = p_1,\ T([0:1:0]) = p_2,\ T([0:0:1]) = p_3,\ T([1:1:1]) = p_4.$$

(ii) *Given five distinct points in* $\mathbb{P}^2$*, no three of which are collinear, show that there is an unique irreducible projective conic passing though all five points. You may want to use part* (i) *to assume that four of the points are* $[1:0:0], [0:1:0], [0:0:1], [1:1:1]$.

Answer: (i) Let $p_i = [a_i : b_i : c_i]$ for $1 \leq i \leq 4$. Define

$$A = \begin{pmatrix} \alpha a_1 & \alpha b_1 & \alpha c_1 \\ \beta a_2 & \beta b_2 & \beta c_2 \\ \gamma a_3 & \gamma b_3 & \gamma c_3 \end{pmatrix},$$

where α, β, γ will be specified later. In fact, we will require that α, β, γ solve the system

$$\alpha a_1 + \beta b_1 + \gamma c_1 = a_4,$$

$$\alpha a_2 + \beta b_2 + \gamma c_2 = b_4,$$

$$\alpha a_3 + \beta b_3 + \gamma c_3 = c_4.$$

A solution exists since the matrix of coefficients

$$B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

is invertible. Indeed, the rows of B are independent. Otherwise, a nontrivial linear relation between the rows would give a line on which the points p_1, p_2, p_3 lie. Thus B is invertible. Now, the system above has the solution

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = B^{-1} \begin{bmatrix} a_4 \\ b_4 \\ c_4 \end{bmatrix}.$$

Note that the same argument shows that A is invertible. Let

$$S : \mathbb{P}^2 \to \mathbb{P}^2$$

be the linear transformation defined by A. Then S is invertible. A direct computation shows

$$S([1:0:0]) = p_1, S([0:1:0]) = p_2, S([0:0:1]) = p_3, S([1:1:1]) = p_4.$$

The proof is completed letting T be the inverse of S.

(ii) After a linear change of coordinates, we may assume that the five points are $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, $[1:1:1]$ and $[u:v:w]$. The equation $f(x,y,z)$ of any conic passing through the first three points can't contain x^2, y^2, z^2, so

$$f(x,y,z) = ayz + bxz + cxy.$$

The remaining two points impose the conditions

$$a + b + c = avw + buw + cuv = 0.$$

Letting

$$A = \begin{pmatrix} 1 & 1 & 1 \\ vw & uw & uv \end{pmatrix},$$

we see that (a, b, c) must be in the null space of A. The rank of A is 2 (the rank cannot be 1 since then the rows would be proportional, hence $u = v = w$ which is not allowed). Therefore, the null space of this matrix is one dimensional, hence the conic passing through the 5 points is unique. The conic cannot be reducible since then it would be union of two lines. One of the lines would have to contain 3 points but that contradicts the general position assumption.

□

Problem 2. *(Twisted curves and complete intersections.)*

A variety Y of dimension r in $\mathbb{P}^n$ is a strict complete intersection if $I(Y)$ can be generated by $n - r$ elements. Y is a set-theoretic complete intersection if Y can be written as the intersection of $n - r$ hypersurfaces.

(i) *Show that a strict complete intersection is a set theoretic complete intersection.*

(ii) *Show that the twisted cubic T in $\mathbb{P}^3$ can be written as the set-theoretic intersection of the quadric and the cubic*

$$Q = \mathcal{Z}(y^2 - xz), C = \mathcal{Z}(z^3 + xw^2 - 2yzw).$$

In particular, T is a set theoretic complete intersection.

(iii) *Show that the twisted cubic in $\mathbb{P}^3$ is the intersections of the three quadrics*

$$Q_1 = \mathcal{Z}(xz - y^2), Q_2 = \mathcal{Z}(xt - yz), Q_3 = \mathcal{Z}(yt - z^2).$$

Show that any two of these quadrics will not intersect in the twisted cubic.

(iv) *Explain that the ideal of the twisted cubic $I(T)$ cannot be generated by two elements, hence T is not a strict complete intersection.*

Answer: (i) Let $f_1, \ldots, f_{n-r}$ be the generators of the ideal $I(Y)$ for a strict complete intersection Y. Then

$$Y = \mathcal{Z}(f_1) \cap \ldots \cap \mathcal{Z}(f_{n-r})$$

exhibits Y as an intersection of $n - r$ hypersurfaces.

(ii) A point in the twisted cubic T has coordinates (t^3, t^2s, ts^2, s^3) so it clearly satisfies the equations of Q and C. Conversely, pick a point in the intersection of Q and C. We compute

$$(xw - yz)^2 = x^2w^2 + y^2z^2 - 2xyzw = x(xw^2 + z^3 - 2yzw) = 0 \implies xw = yz$$

$$(yw - z^2)^2 = y^2w^2 + z^4 - 2ywz^2 = xzw^2 + z^4 - 2ywz^2 = z(xw^2 + z^3 - 2yzw) = 0 \implies yw = z^2.$$

By (iii), we know that such a point belongs to the twisted cubic.

(iii) The twisted cubic is given by

$$[x : y : z : t] = [a^3 : a^2b : ab^2 : b^3]$$

for some $[a : b] \in \mathbb{P}^1$. Clearly, points of these type satisfy the equations

$$xz = y^2, xt = yz, yt = z^2,$$

so the twisted cubic is contained in the intersection $Q_1 \cap Q_2 \cap Q_3$.

Conversely, given a point $p = [x : y : z : t]$ in the intersection of the three quadrics Q_1, Q_2, Q_3, set

$$[a : b] = [x : y]$$

if $x \neq 0$. Otherwise, if $x = 0$, then $y = z = 0$ and set $[a : b] = [0 : 1]$. We claim

$$[x : y : z : t] = [a^3 : a^2 b : ab^2 : b^3]$$

e.g. p lies on the twisted cubic. This is clear for the case $x = 0$. If $x \neq 0$, then $a \neq 0$, and from the first and third equations we have

$$y = x \cdot \frac{b}{a}.$$

Using the first and second equations, we solve

$$z = x \cdot \frac{b^2}{a^2}, t = x \cdot \frac{b^3}{a^3}.$$

Then,

$$[x : y : z : t] = \left[x : x\frac{b}{a} : x\frac{b^2}{a^2} : x\frac{b^3}{a^3}\right] = [a^3 : a^2 b : ab^2 : b^3].$$

Finally, note that $[0 : 0 : 1 : 0] \in Q_1 \cap Q_2$ but not in Q_3; $[0 : 1 : 0 : 0] \in Q_2 \cap Q_3$ but not in Q_1; $[1 : 0 : 0 : 1] \in Q_3 \cap Q_2$ but not in Q_1. So any of the two quadrics can't generate the twisted cubic.

(iv) It is clear that $I(T)$ cannot contain linear elements $ax + by + cz + dw$ because then T would be contained in a hyperplane $ax + by + cz + dw = 0$ which would mean

$$as^3 + bs^2 t + cst^2 + ds^3 = 0$$

for all s, t which clearly is impossible. Moreover, we have seen in (iii) that the ideal of T contains 3 independent elements of degree 2. Since the space of homogeneous elements in $I(T)$ of degree 2 is 3 dimensional, any set of generators for $I(T)$ has to have at least 3 elements.

□

Problem 3. *(Introduction to moduli theory.) Show that for any 3 lines L_1, L_2, L_3 in $\mathbb{P}^3$, there is a quadric $Q \subset \mathbb{P}^3$ containing all three of them.*

(i) *First, observe that any homogeneous degree 2 polynomial in 4 variables has 10 coefficients. These coefficients can be regarded as a point in the projective space $\mathbb{P}^9$. Show that this point only depends on the quadric Q and not on the polynomial defining it. Let us denote this point by p_Q. Show that any point $p \in \mathbb{P}^9$ determines a quadric in $\mathbb{P}^3$.*

(ii) *Consider a line $L \subset \mathbb{P}^3$. Show that there is a codimension 3 projective linear subpace*

$$H_L \subset \mathbb{P}^9$$

such that

$$L \subset Q \text{ iff and only if } p_Q \in H_L.$$

(iii) *Show that any three codimension 3 projective linear subspaces of $\mathbb{P}^9$ intersect. In particular, show that*

$$H_{L_1} \cap H_{L_2} \cap H_{L_3} \neq \emptyset,$$

and conclude that L_1, L_2, L_3 are contained in a quadric Q.

Answer: (i) Consider a homogeneous degree 3 polynomial

$$F(X_0, X_1, X_2, X_3) = a_0X_0^2 + a_1X_0X_1 + \cdots + a_{10}X_3^2.$$

This has 4 square terms and $\binom{4}{2} = 6$ mixed terms, i.e. 10 terms in total. We associate the quadric $Q := \mathcal{Z}(F)$ the point point

$$p_Q = [a_0 : a_1 : \ldots : a_{10}] \in \mathbb{P}^9.$$

Note that this association is independent of the defining equation of the quadric. Indeed, if F and G define the same quadric, then by the Nullstellensatz it follows that $G = cF$ for some constant $c \neq 0$. But then the point p_Q does not change, since the coefficients a_i are considered projectively. Finally, this association is clearly bijective, since for any $p_Q \in \mathbb{P}^9$ we can recover the equation of the quadric Q (up to scalars).

(ii) Assume that the line L is given by $X_0 = X_1 = 0$. Then

$$L \subset Q \cong F(0, 0, X_2, X_3) = 0.$$

Since

$$F(0, 0, X_2, X_3) = a_8X_2^2 + a_9X_2X_3 + a_{10}X_3^2,$$

we obtain

$$a_8 = a_9 = a_{10} = 0.$$

These equations define a codimension 3 projective space $H_L \subset \mathbb{P}^9$ such that

$$L \subset Q \text{ if and only if } p_Q \in H_L.$$

If L' is any line and Q' is a quadric, we can change coordinates linearly so that L' becomes the line L: $X_0 = X_1 = 0$. After the coordinate change, Q' is mapped to another quadric Q and the coefficients of Q are linear combinations of coefficients of Q'. From above

$$L \subset Q \text{ if and only if } p_Q \in H_L.$$

Note that

$$L' \subset Q' \text{ if and only if } L \subset Q.$$

Therefore there exists a codimension 3 projective space $H_{L'} \subset \mathbb{P}^9$, the transformation of H_L via the change of coordinates, such that

$$L' \subset Q' \text{ if and only if } p_{Q'} \in H_{L'}.$$

(iii) The codimension 3 subspaces H_L correspond to 7 dimensional linear subspaces $A_L \subset \mathbb{A}^{10}$ cut out by the same 3 linear equations. Now,

$$A_{L_1} \cap A_{L_2} \cap A_{L_3}$$

is described by 9 linear equations in $\mathbb{A}^{10}$. Such a system of equations must have a nontrivial solution p. This solution p considered projectively satisfies

$$p \in H_{L_1} \cap H_{L_2} \cap H_{L_3}.$$

Now, p determines a quadric Q with $p = p_Q$. It follows by (iii) that

$$L_1 \cap L_2 \cap L_3 \subset Q.$$

□

Problem 4. *We will make the space of all lines in $\mathbb{P}^n$ into a projective variety. We define a set-theoretic map*

$$\phi : \{\text{lines in } \mathbb{P}^n\} \to \mathbb{P}^N$$

with

$$N = \binom{n+1}{2} - 1$$

as follows. For every line $L \subset \mathbb{P}^n$, choose two distinct points

$$P = (a_0 \dots a_n) \text{ and } Q = (b_0 \dots b_n)$$

on L and define $\phi(L)$ to be the point in $\mathbb{P}^N$ whose homogeneous coordinates are the maximal minors of the matrix

$$\begin{pmatrix} a_0 : \dots : a_n \\ b_0 : \dots : b_n \end{pmatrix}$$

in any fixed order. Show that:

(i) *The map ϕ is well-defined and injective. The map ϕ is called the Plucker embedding.*

(ii) *The image of ϕ is a projective variety that has a finite cover by affine spaces $\mathbb{A}^{2(n-1)}$. You may want to recall the Gaussian algorithm which brings almost any matrix as above into the form*

$$\begin{pmatrix} 1 \ 0 \ a'_2 \dots a'_n \\ 0 \ 1 \ b'_2 \dots b'_n \end{pmatrix}.$$

(iii) *Show that $G(1,1)$ is a point, $G(1,2) = \mathbb{P}^2$, and $G(1,3)$ is the zero locus of a quadratic equation in $\mathbb{P}^5$.*

Answer: (i) Let $e_0, \dots, e_n$ be the standard basis of the vector space $V = k^{n+1}$. A line L corresponds to a 2-dimensional subspace in k^{n+1}, also denoted by L. We claim that the map ϕ can be described as follows. Picking a basis v, w for the subspace,

$$\phi(L) = [v \wedge w] \in \mathbb{P}(\Lambda^2 V) \cong \mathbb{P}^N.$$

Indeed, if $P = (a_0 : \dots : a_n)$ and $Q = (b_0 : \dots : b_n)$ are two distinct points, then we may take

$$v = \sum a_i e_i, w = \sum b_i e_i.$$

Thus

$$v \wedge w = \sum_{i,j} a_i b_j e_i \wedge e_j = \sum_{i<j} (a_i b_j - a_j b_i) e_i \wedge e_j.$$

Therefore, in the basis $e_i \wedge e_j$, the coordinates are the 2×2-minors of the matrix in (i). This shows that ϕ is well-defined.

To check injectivity, let L' be another line corresponding to a 2-dimensional subspace. If $L' \cap L = 0$, then pick a basis v_0, v_1, v_2, v_3 for $L \oplus L'$ with

$$v_0, v_1 \in L, v_2, v_3 \in L',$$

and extend it to a basis $v_0, \dots, v_n$ of $V = k^{n+1}$. Thus $v_i \wedge v_j$ for $i < j$ is a basis for $\Lambda^2 V$. In particular, $v_0 \wedge v_1$ and $v_2 \wedge v_3$ are not multiples, or equivalently

$$\phi(L) \neq \phi(L').$$

The same argument applies if L and L' have a 1 dimensional intersection.

(ii) To show projectivity, we will prove that

$$\omega \in \Lambda^2 V \text{ splits as } \omega = v \wedge w \text{ if and only if } \omega \wedge \omega = 0.$$

In particular, if

$$\omega = \sum \omega_{ij} e_i \wedge e_j,$$

then

$$\omega \wedge \omega = \sum_{i<j,k<l} \omega_{ij}\omega_{kl} e_i \wedge e_j \wedge e_k \wedge e_l = \sum_{i<j<k<l} (\omega_{ij}\omega_{kl} - \omega_{ik}\omega_{jl} + \omega_{il}\omega_{jk}) e_i \wedge e_j \wedge e_k \wedge e_l$$

so the image of ϕ is cut by the quadrics

$$\omega_{ij}\omega_{kl} - \omega_{ik}\omega_{jl} + \omega_{il}\omega_{jk} = 0.$$

We now prove the claim. It is clear that if $\omega = v \wedge w$ then $\omega \wedge \omega = 0$. Conversely, we will induct on n, the base case $n = 2$ being clear. Let us write

$$\omega = e_0 \wedge \eta + \omega'$$

where ω', η do not contain the vector e_0. Thus

$$0 = \omega \wedge \omega = 2e_0 \wedge \eta \wedge \omega' + \omega' \wedge \omega'.$$

This implies that

$$\omega' \wedge \omega' = 0$$

hence by induction

$$\omega' = v \wedge w,$$

with v, w being in the subspace spanned by $e_1, \ldots, e_n$. Also, we know

$$e_0 \wedge \eta \wedge \omega' = 0 \implies \eta \wedge v \wedge w = 0.$$

This shows that η cannot be independent of v, w hence

$$\eta = av + bw.$$

Collecting terms we find

$$\omega = e_0 \wedge (av + bw) + v \wedge w = (v + be_0) \wedge (w + ae_0)$$

as claimed.

For the last part, note that one of the coordinates of $\phi(L)$ must be non-zero. Without loss of generality let us assume it is the coordinate corresponding to $e_0 \wedge e_1$. This means that the first 2×2 minor of the matrix

$$\begin{pmatrix} a_0 : \ldots : a_n \\ b_0 : \ldots : b_n \end{pmatrix}$$

is non-zero. The Gaussian algorithm brings this matrix into the form

$$\begin{pmatrix} 1 \ 0 \ a'_2 \ldots a'_n \\ 0 \ 1 \ b'_2 \ldots b'_n \end{pmatrix}.$$

The association

$$L \to (a'_2, \ldots, a'_n, b'_2, \ldots, b'_n) \in \mathbb{A}^{2(n-1)}$$

shows how to cover the image of ϕ by affine opens isomorphic to $\mathbb{A}^{2(n-1)}$.

(iii) All these statements are particular cases of what we proved in part (ii). For instance, to see that $G(1, 3)$ is a quadric in $\mathbb{P}^5$ given by

$$x_0x_5 - x_1x_4 + x_2x_3 = 0.$$

□