Math 100A, Problem Set 3. Due Friday October 25.

1. Solve Problem 21 (a)(b), Section 1.4.

2. Solve the following modified version of Problem 26, Section 1.4, which is a followup of Problem 4 on the Midterm.
   (i) If \( \gamma = (a_1 a_2 \ldots a_\ell) \) is a cycle of length \( \ell \), show that
   \[ \tau \gamma \tau^{-1} = (\tau(a_1) \, \tau(a_2) \ldots \tau(a_\ell)). \]
   (ii) Show that for any cycles \( \gamma_1, \gamma_2 \) of length \( \ell \) we can find a permutation \( \tau \) such that
   \[ \tau \gamma_1 \tau^{-1} = \gamma_2. \]
   We say that any two cycles of length \( \ell \) are conjugate.

3. (i) For \( 1 < i < j \leq n \) show the following identity holds in \( S_n \):
   \[ (i \, j) = (1 \, i)(1 \, j)(1 \, i). \]
   (ii) Show that any permutation in \( S_n \) can be written as product of transpositions chosen among
   \( (1 \, 2), (1 \, 3), \ldots, (1 \, n) \).

4. (i) Show that any permutation in \( S_n \) can be written as product of transpositions chosen among
   \( (1 \, 2), (2 \, 3), \ldots, (n - 1 \, n) \).
   (ii) Let
   \[ \sigma = (1 \, 2 \ldots n). \]
   Use 3(ii) to show that
   \[ \sigma(i - 1 \, i) \sigma^{-1} = (i \, i + 1). \]
   Conclude from (iii) that any permutation in \( S_n \) can be written as product of the three permutations \( \sigma, \sigma^{-1}, \tau = (1 \, 2) \).

5. Section 2.2, Problem 1.

6. Section 2.2, Problem 7. The group introduced in this problem is called the Heisenberg group named after the physicist Werner Heisenberg.
7.

(i) Let $G$ denote the set of matrices

$$ G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \text{ where } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}. $$

Show that $G$ is a group under matrix multiplication. This group is denoted $SL_2(\mathbb{Z})$.

*Hint:* The issue here is to observe that inverses have integer entries. You should write down the inverse matrix explicitly.

(ii) Let $n$ be an integer, and let $G$ denote the set of matrices

$$ G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \text{ where } a, b, c, d \in \mathbb{Z}_n \text{ and } ad - bc = 1 \text{ in } \mathbb{Z}_n \right\}. $$

Show that $G$ is a group under matrix multiplication. This group is denoted $SL_2(\mathbb{Z}_n)$.

(iii) Let $G$ denote the set of matrices

$$ G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \text{ where } a, b, c, d \in \mathbb{Z}_2 \text{ and } ad - bc = 1 \text{ in } \mathbb{Z}_2 \right\}. $$

List all elements of $G$. Show that $G$ has 6 elements.