1. (*The symmetric group* $S_3$.) This problem asks you to fill in some details of Example 8, page 78.

In the symmetric group $S_3$, consider the cycles

$\sigma = (1 \ 2 \ 3), \ \tau = (1 \ 2)$. 

(i) Show that all elements of $S_3$ are

$S_3 = \{\epsilon, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$. 

(ii) Show that

$\sigma^3 = \epsilon, \ \tau^2 = \epsilon, \ \sigma\tau\sigma = \tau$. 

(iii) Show that $S_3$ is not abelian.

2. (*Squares and elements of order 2.*) Solve the following modified version of Problems 5 and 6, Section 2.3.

(i) Show that if $G$ is abelian, then

$H = \{g^2 : g \in G\}$

is a subgroup of $G$.

(ii) Give an example showing that the converse of (i) is false.

(iii) If $G$ is abelian, show that

$K = \{g \in G, \ g^2 = 1\}$

is a subgroup of $G$.

(iv) Give an example of a group $G$ such that $K$ is not a subgroup.

*Remark:* Your examples in (ii) and (iv) should be noncommutative. In fact, you can use the same group for both (ii) and (iv).

3. (*Centralizer.*) Let $G$ be a group, and let $g \in G$. Define

$C(g) = \{h \in G : hg = gh\}$. 

This is called the centralizer of $g$ and consists in elements that commute with $g$. Show that $C(g)$ is a subgroup of $G$.

4. Solve Problem 8(a), Section 2.3. Read Problem 7(a) and note that it is in fact a particular case of Problem 8(a).

5. (*Subgroups.*) Solve Problem 11, Section 2.3.
6. *(Center of the matrix group.)* Solve Problem 22, Section 2.3, asking you to find the center of the group $GL_2(\mathbb{R})$.

*Hint:* Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in the center, and write down explicitly the condition that $AB = BA$

for all matrices $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in GL_2(\mathbb{R})$.

7. *(Center of the alternating group.)* In Example 9, page 88, it is shown that the center of the group $S_n$ is trivial, for $n \geq 3$.

Using a similar idea, prove that the center of $A_n$ is trivial for $n \geq 4$.

*Hint:* Let $\sigma$ be in the center. Let $\sigma(a) = b \neq a$. Pick $c$ and $d$ distinct from each other, and distinct from $a, b$. Let $\tau = (bcd)$. Do $\sigma$ and $\tau$ commute?

8. Section 2.3, Problem 17.

9. *(Order of elements.)* Section 2.4, solve Problem 1(b), 7, 10, 11.