Math 100A - Fall 2019 - Practice Problems for Final Exam

The midterm will cover Chapters 1 and 2 in the book. The main topics are:

- equivalence relations, congruences, \( Z_n \), invertible elements, Fermat’s theorem, Euler’s theorem, Chinese remainder theorem, Wilson’s theorem
- permutations, cycles, transpositions, parity of permutations, order of permutations, symmetric group, alternating group
- groups, subgroups, center, centralizer, product groups
- cyclic groups, order of elements, properties of order, Lagrange’s theorem
- homomorphisms, kernel, image, isomorphisms, automorphisms, inner automorphisms
- normal subgroups, cosets, quotient groups
- isomorphism theorems
- classification of groups with few elements, Klein group, octonionic group, groups with \( p, 2p, p^2 \) elements, dihedral group.

0. Please make sure to review the definitions and all proofs covered in class. You may be asked to define terms or prove a statement which is similar to theorems proved in class.

Also please review the homework problems.

1. (Congruences. Fermat. Chinese remainder theorem.) Let \( x = 2^{2019} \).
   (i) Find \( x \mod 17 \) and \( x \mod 19 \) using Fermat’s theorem.
   (ii) Use part (i) and the Chinese remainder theorem to find \( x \mod 17 \cdot 19 \).

2. (Permutations.)

   Consider the permutation
   \[
   \sigma = \begin{pmatrix}
   1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
   4 & 3 & 6 & 5 & 1 & 2 & 9 & 10 & 8 & 7 
   \end{pmatrix}
   \]
   Determine the parity of \( \sigma \) and the order of \( \sigma \) in \( S_{10} \).

3. (Permutations.) Solve the following version of Problem 25, page 66.
   (i) Consider two disjoint transpositions \( (ab) \) and \( (cd) \) in \( S_n \). Write \( (ab)(cd) \) as product of two cycles of length 3.
   (ii) Using (i) show that the product of two transpositions \( (ab)(cd) \) is always a product of cycles of length 3. You need to address here the case when the transpositions are not disjoint.
   (iii) Show that any even permutation is product of cycles of length 3.

4. (Lagrange’s theorem. Klein group.) Let \( G \) be a group of order 10. Show that \( G \) contains an element of order 5.

   \textit{Hint:} This is a part of the argument we used to classify groups with \( 2p \) elements.
5. (Lagrange’s theorem. Order of powers of elements.) Solve problem 23, page 117 in the book: if \( G \) is a group of order \( p^n \), show that \( G \) admits an element of order exactly \( p \).

6. (Subgroups. Normal subgroups.)

(i) Show that in general \( HK \) may not be a subgroup of \( G \). You may wish to take \( G \) to be \( S_3 \) and \( H, K \) subgroups generated by two transpositions of your choice.

(ii) Show that if \( H, K \) are subgroups of \( G \) such that \( HK = KH \) then \( HK \) is a subgroup of \( G \).

(iii) Show that if \( H \) is normal or if \( K \) is normal, then \( HK = KH \) and thus by (ii) \( HK = KH \) is a subgroup of \( G \).

Parts (ii) and (iii) were proved in class as part of the second isomorphism theorem, but here I ask you to review the argument.

(iv) Show that if \( H, K \) are normal, then \( HK \) is a normal subgroup of \( G \).

7. (Dihedral group.) Solve the following version of Problem 29, page 137. Let \( k, n \) be two integers with \( n = kq \). Consider the dihedral group \( D_n \) with generators \( a, b \) such that

\[
o(a) = n, o(b) = 2, aba = b.
\]

(i) Let \( K = \langle a^k \rangle \) be the cyclic subgroup of \( D_n \). Show that \( K \simeq C_q \), the cyclic group of order \( q \).

(ii) Consider the quotient group \( D_n/K \). Show that there are elements \( \alpha, \beta \) in the quotient group \( D_n/K \) such that

\[
o(\alpha) = k, o(\beta) = 2, \alpha \beta \alpha = \beta.
\]

Conclude that

\[D_n/K \simeq D_k.\]

You may wish to take \( \alpha = aK \) and \( \beta = bK \).

8. (Normal subgroups. Automorphisms.) Let \( f : G \rightarrow G \) be an automorphism. Let \( H \) be a subgroup of \( G \).

(i) Show that \( f(H) \) is also a subgroup of \( G \).

(ii) In particular, picking \( f \) to be an inner automorphism conclude that \( gHg^{-1} \) is a subgroup of \( G \).

(iii) Let \( G \) be a group admitting a unique subgroup \( H \) of order \( k \). Show that \( H \) is normal.

9. (Inner automorphisms. First isomorphism theorem.) Show that \( G/Z(G) \simeq \text{Inn}(G) \). You may wish to find the kernel of the natural map

\[G \rightarrow \text{Inn}_G, \ a \mapsto \sigma_a.\]

If you get stuck, this is Theorem 5, page 140.
10. (*Groups with p elements. Factor groups.*) Solve Problem 14, page 136: if $K$ is a normal subgroup of $G$ of index $p$, $p$ prime, show that there exists $g \in G$ such that

$$G = K \cup gK \cup \ldots \cup g^{p-1}K.$$ 

11. (*Lagrange’s theorem. Isomorphisms.*) Solve Problem 22, page 130. If $H, K$ are subgroups of $G$ such that

(i) $hk = kh$ for all $k \in K, h \in H$,

(ii) $|G| = mn, |K| = m, |H| = n$ and $\gcd(m, n) = 1$.

Show that

$$G \cong K \times H.$$