Math 100A - Fall 2019 - Practice Problems for Midterm II

The midterm will cover Chapters 2.1 - 2.5 in the book. The main topics are:

- binary laws, groups, basic properties, examples
- subgroups, center, centralizer, normal subgroups
- order of elements, properties of order, order of permutations, cyclic groups, subgroups of cyclic group
- homomorphisms, isomorphisms, examples, basic properties, kernel, image
- automorphisms, inner automorphisms

1. Please make sure to review the definitions and all proofs covered in class. You may be asked to define terms or prove a statement which is similar to theorems proved in class.

Also please review the homework problems.

2. (Binary laws, groups, isomorphisms.) Consider $G = \mathbb{Q} \setminus \{-1\}$ endowed with the binary law $a \star b = ab + a + b$.
   (i) Show that $\star$ is a well-defined binary law on $G$.
   (ii) Show that $(G, \star)$ is a group with 0 as identity.
   (iii) Show that $(G, \star)$ is isomorphic to $(\mathbb{Q} \setminus \{0\}, \cdot)$

3. (Groups.) Show that if $G$ is a group with an even number of elements, there exists $a \in G$, $a \neq e$ such that $a^2 = e$. You may wish to group elements in pairs $(a, a^{-1})$.

4. (Subgroups, normal subgroups.) Let $H$ be a subgroup of a group $G$. The normalizer of $H$ in $G$ is defined as
   \[ N_G(H) = \{ g \in G : gHg^{-1} = H \}. \]
   (i) Show that $N_G(H)$ is a subgroup of $G$.
   (ii) Show that $H$ is normal if and only if $N_G(H) = G$.

5. (Subgroups.) If $X$ is a subset of a group $G$, define
   \[ \langle X \rangle = \{ x_1^{k_1} \cdots x_m^{k_m} : x_i \in X, k_i \in \mathbb{Z}, m \geq 1 \}. \]
   (i) Show that $\langle X \rangle$ is a subgroup of $G$.
   (ii) Show that if $G = \langle X \rangle$ and $xy = yx$ for all $x, y \in X$ then $G$ is abelian.

6. (Normal subgroups.) Show that $H = \{1, \sigma, \sigma^2\}$ is a normal subgroup of $G = S_3$. There are two ways of solving this problem. One involves direct verification (which is tedious and I won’t recommend).
A second method is based on a general argument. If \( g \in G \setminus H \), show that \( H \) and \( gH \) are disjoint. Similarly show that \( H \) and \( Hg \) are disjoint. Comparing the number of elements of \( G \), \( H \), \( gH \) and \( Hg \) conclude that \( Hg = gH \) so that \( H \) is normal.

7. (Order of permutations.) Find the smallest integer \( n \) such that \( \sigma^n = \epsilon \) for all \( \sigma \in S_6 \).

8. (Cyclic groups and their subgroups.) Let \( G = \langle g \rangle \) be a finite cyclic group of order \( n \). Show that if \( a \mid n \) and \( b \mid n \) then \[
\langle g^a \rangle \cap \langle g^b \rangle = \langle g^c \rangle
\]
where \( c \) is the least common multiple of \( a, b \).

9. (Order of elements.) Let \( g \in G \) be an element of order \( m \), and let \( h \in H \) be an element of order \( n \). Find the order of the element \( (g, h) \) in \( G \times H \).

10. (Automorphisms.) If \( G \) is an infinite cyclic group, find \( \text{Aut}(G) \).

11. (Isomorphisms. Order.) Exhibit, with proof, three nonisomorphic groups of order 27.

12. (Cyclic groups, their subgroups. Automorphisms.)
   (i) For \( n = p_1^{a_1} \cdots p_k^{a_k} \) with \( p_i \) prime, find the number of subgroups of \( C_n \).
   (ii) Draw the lattice of subgroups of \( C_{p^2q^2} \), where \( p, q \) are prime.
   (iii) Find the number of automorphisms of \( C_{12} \).

13. (Inner automorphisms.) Show that if \( Z(G) = \{1\} \), then \( G \simeq \text{Inn}(G) \).